上州無智亦無大
剛毅木訥易被欺
唯以三直接萬人
至誠依神期勝利
鑑

Introduction to
Inflationary Cosmology

東京大学 大学院
理学系研究科·理学部
SCHOOL OF SCIENCE, THE UNIVERSITY OF TOKYO

RESEARCH CENTER FOR THE EARLY UNIVERSE
THE UNIVERSITY OF TOKYO
RESCEU

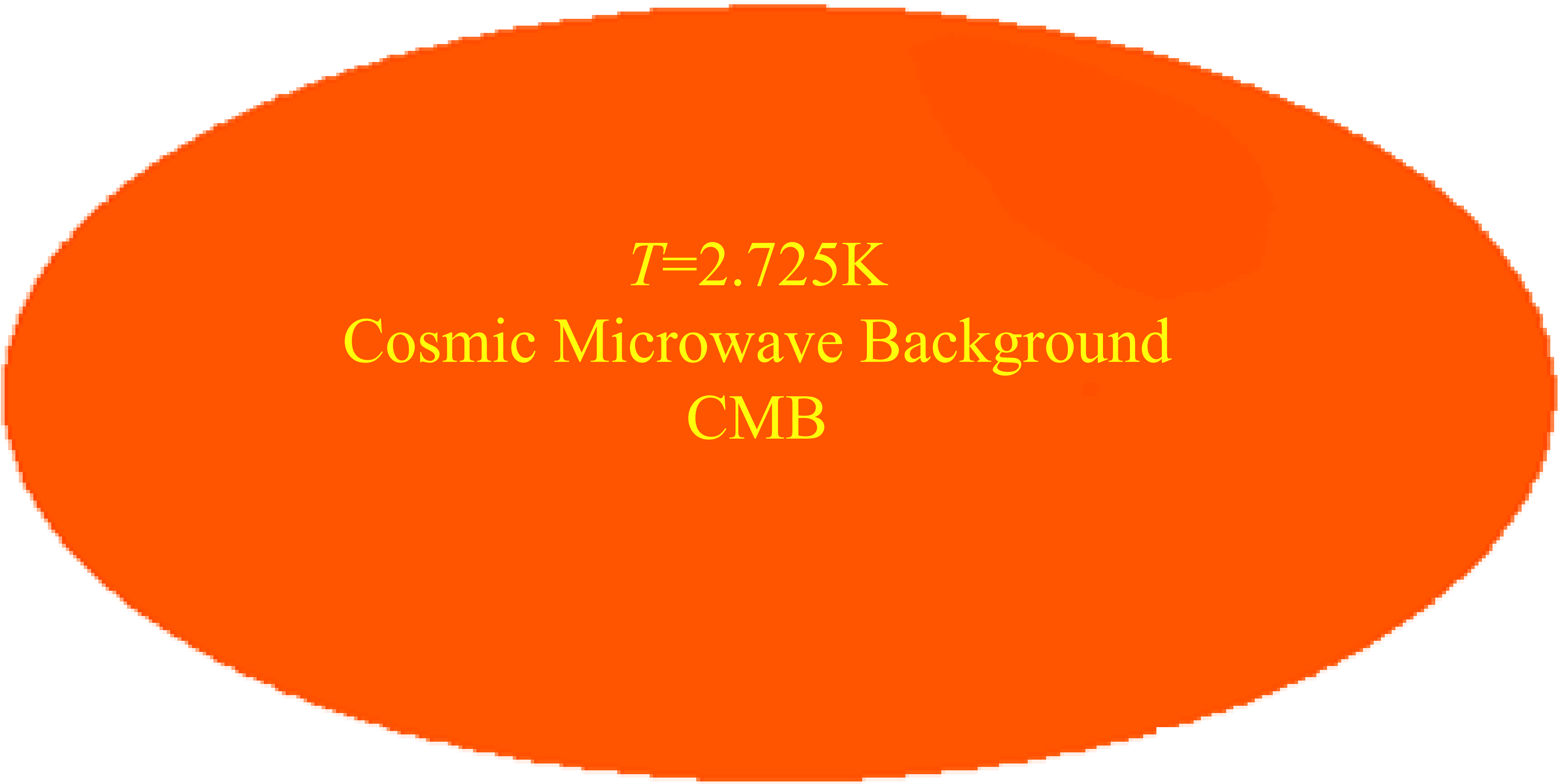
横山順一
Jun'ichi Yokoyama
(RESCEU&IPMU, Tokyo)

Introduction to

Inflationary Cosmology

These lectures are primarily intended for
Those who have never studied inflation
as well as for
Those who have studied inflation
but are working on bouncing cosmology
without inflation.

Full sky map of microwave background radiation #1



The Universe is globally isotropic and homogeneous

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

Scale factor

Curvature

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

$$H = \frac{\dot{a}}{a}$$ Hubble parameter

$$\Omega = \frac{\rho}{\rho_{cr}} = \frac{8\pi G}{3H^2}\rho$$ Density parameter

$\Omega_m, \quad \Omega_b$

$$\frac{K}{a^2} = (\Omega + \Omega_\Lambda - 1)\, H^2$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$ cosmological constant (dark energy)

Energy momentum tensor: Perfect Fluid

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho + P)u^\mu u^\nu \qquad u^\mu = (\gamma, \gamma \mathbf{v}/a)$$

Pressure (P)　Energy density (ρ)

Conservation

$$T^\mu_{\nu\ ;\mu} = 0 \quad \Rightarrow \quad \underbrace{\frac{d}{dt}\rho a^3 = -P\frac{da^3}{dt}}_{dE = -PdV + d'Q} \quad \Rightarrow \quad \frac{d\rho}{dt} + 3H(\rho + P) = 0$$

$d'Q = 0$　　$d'Q = TdS = 0$

In quasi-static processes

Comoving entropy is conserved unless some nonequilibrium processes take place.

The Einstein equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

The Particle Horizon and The Hubble Horizon

★ Particle Horizon (Physics Horizon) $d_H(t)$

The maximum length causal interaction can reach by the time t
= The maximum length light can travel by the time t.
Light travels with $ds^2 = -dt^2 + a^2(t)d\chi^2 = 0.$

$$d_H(t) = a(t)\int d\chi = a(t)\int_{t_i}^{t}\frac{dt'}{a(t')} = \begin{cases} \boxed{\dfrac{t}{1-m}\left[1-\left(\dfrac{t}{t_i}\right)^{m-1}\right] \cong \dfrac{t}{1-m} \quad a(t)\propto t^m \text{ with } m<1 \text{ (Matter or Rad'n era)}} \\ \dfrac{t}{m-1}\left[\left(\dfrac{t}{t_i}\right)^{m-1} -1\right] \sim t^m \propto a(t) \quad a(t)\propto t^m \text{ with } m>1 \text{ (Accelerated Expansion)} \\ \dfrac{1}{H}\left(e^{H(t-t_i)}-1\right) \sim e^{Ht} \propto a(t) \quad a(t)\propto e^{Ht} \text{ (Exponential Expansion)} \end{cases}$$

The Classical Big Bang Theory has only this epoch.

Hubble Horizon, Hubble Radius, Hubble Length

- The scale causal interaction is possible within the cosmic expansion time H^{-1}

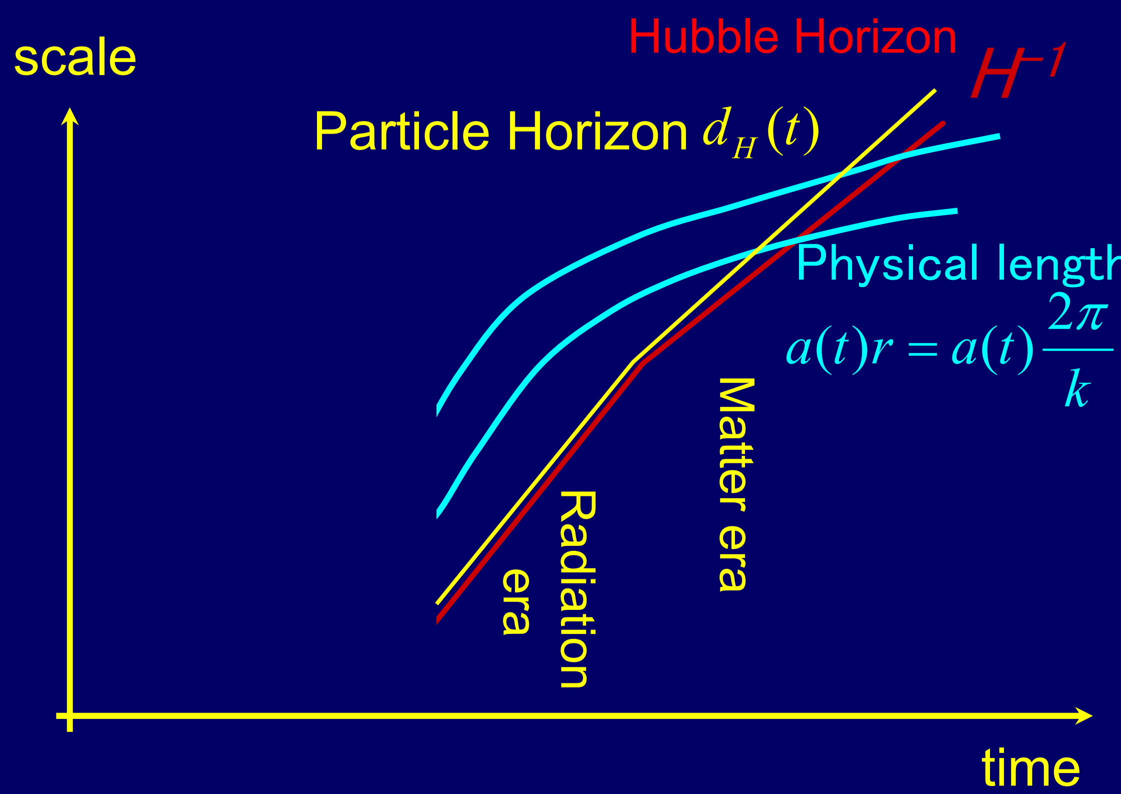
- One can neglect effects of expansion within this time scale, so one finds

$$cH^{-1} = H^{-1} = \begin{cases} \dfrac{t}{m} & a(t) \propto t^{m} \\ H^{-1} & a(t) \propto e^{Ht} \end{cases}$$

- In the expanding Universe, various events have taken place at different epochs of the relevant energy scales. The Hubble radius gives the maximum scale that each event can occur coherently.

- Important when particle physics is applied to cosmology.
- The term “Horizon” most likely means the Hubble horizon.
- The maximum scale we can directly observe at each time.

Evolution of scales in the Classical Big Bang Theory



- In the Classical Big Bang Theory, both the particle horizon and the Hubble horizon evolves in proportion to time, namely more rapidly than the physical length of each coordinate scale $(\propto a(t))$.
- The scales of no previous causal interaction enter the Hubble radius continuously and can be seen for the first time.

They look all the same!＝The Horizon Problem

The Universe at the Decoupling Epoch

The Hubble Radius Then
$\sim$1 angular degree

We must sum up more than 10^5 causal patches to make up the current Hubble volume.

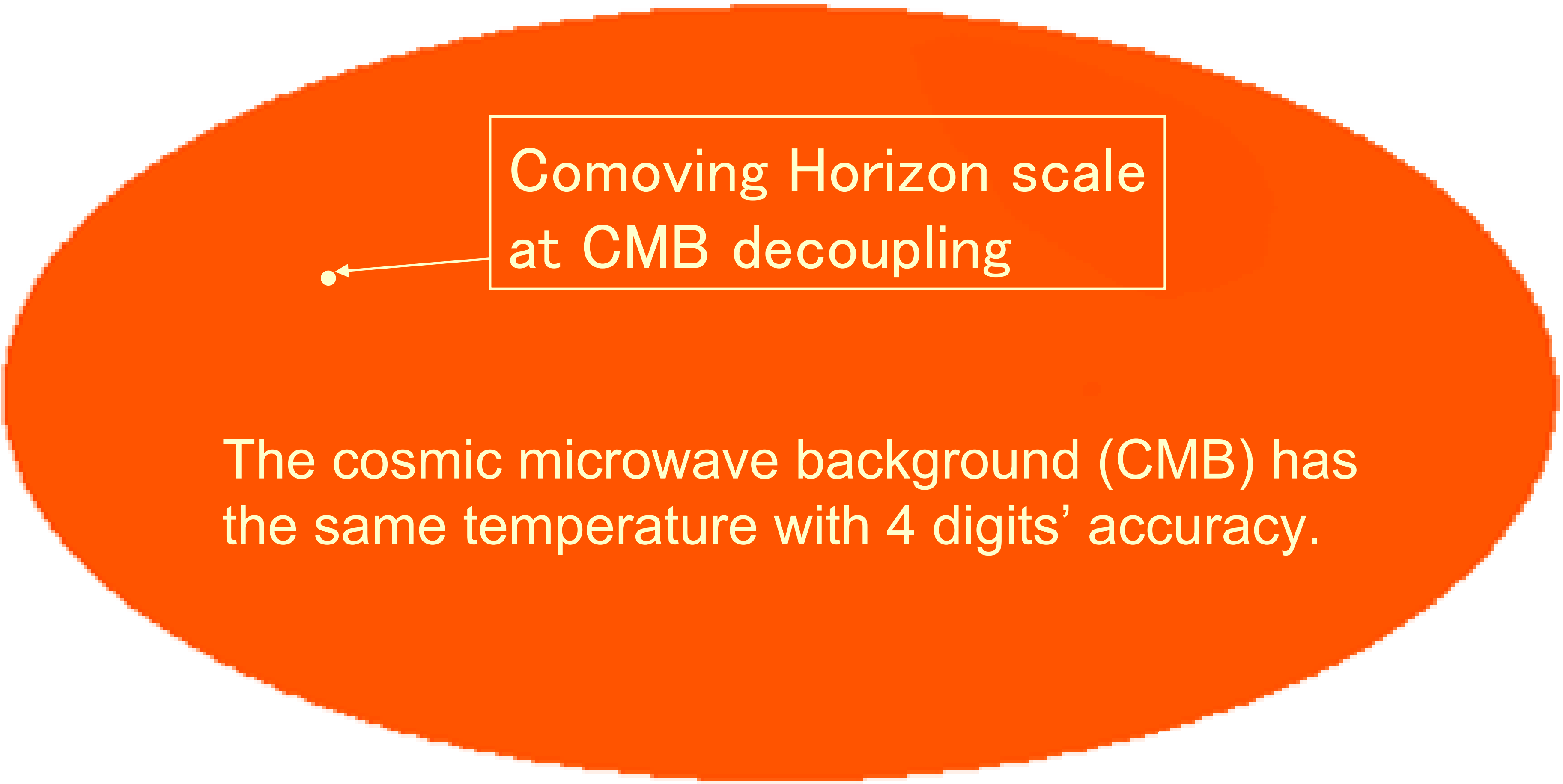
The Horizon Problem

INFLATION in the early Universe

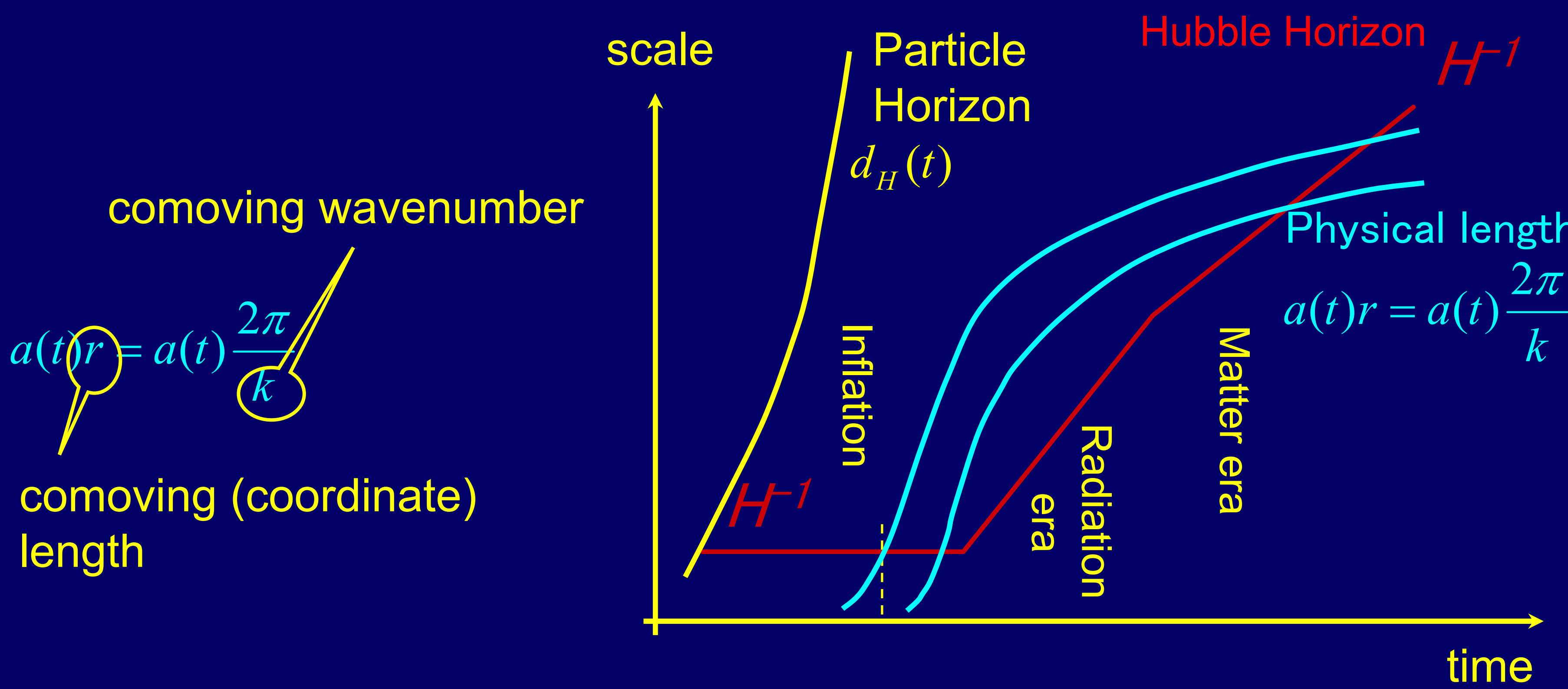
solves....

The Horizon problem

The Universe observed by COBE



Evolution of scales in the Inflationary Cosmology



- ⋆ The particle horizon is exponentially stretched.
- ⋆ Each coordinate scale crosses the Hubble horizon twice, during and after inflation.
- ⋆ In between two horizon crossing epochs, that scale is beyond the Hubble radius and hence invisible.

INFLATION in the early Universe

solves....

The Horizon problem
The Flatness problem

FLATNESS PROBLEM

$$\left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

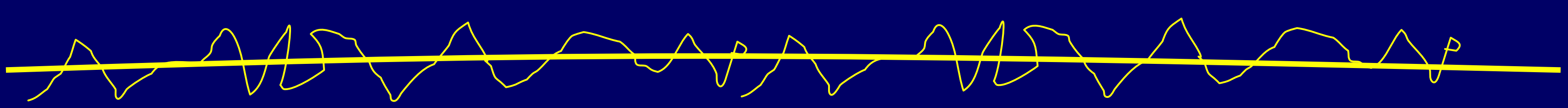
The curvature term decreases less rapidly than matter or radiation.

$\propto a^{-3}$ $\qquad$ $\propto a^{-4}$

$$\frac{K}{a^2} = \left(\Omega_{tot} - 1\right) H^2 \quad \Rightarrow \frac{\Omega_{tot}(t_0) - 1}{\Omega_{tot}(t_i) - 1} = \left(\frac{a(t_i)}{a_0}\right)^2 \left(\frac{H(t_i)}{H_0}\right)^2 \sim \left(\frac{T_0}{T_i}\right)^2 \left(\frac{t_0}{t_i}\right)^2$$

$$\sim 10^{58} \ @\ t_i = t_{Pl}$$

- At the Planck time the curvature radius must have been larger than the Hubble radius by more than 10^{29} times.



INFLATION in the early Universe

solves....

The Horizon Problem
The Flatness Problem
The monopole & other relic Problems

If one monopole is created per horizon@GUT phase transition,

$$\frac{n_M}{s} \approx 10 \frac{M}{M_{Pl}} \approx 10^{-10} \frac{M}{10^{16}\,\mathrm{GeV}} \quad \text{vs current constraint} \quad \frac{n_M}{s} < 10^{-24} \left(\frac{M}{10^{16}\,\mathrm{GeV}} \right)^{-1}$$

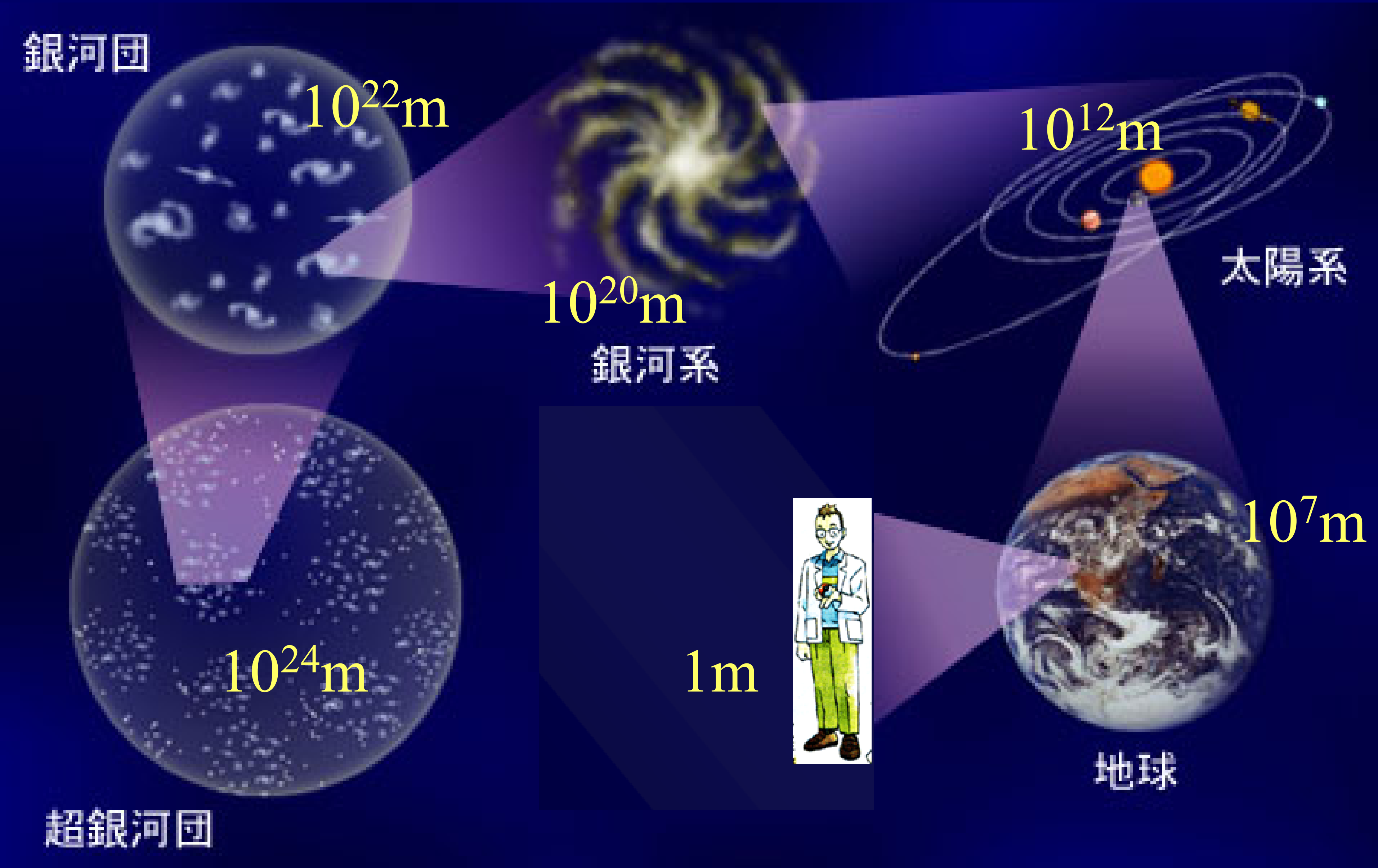
Monopoles and other relics /entropy are NOT diluted by inflationary expansion but by the subsequent entropy production at the reheating.

INFLATION in the early Universe

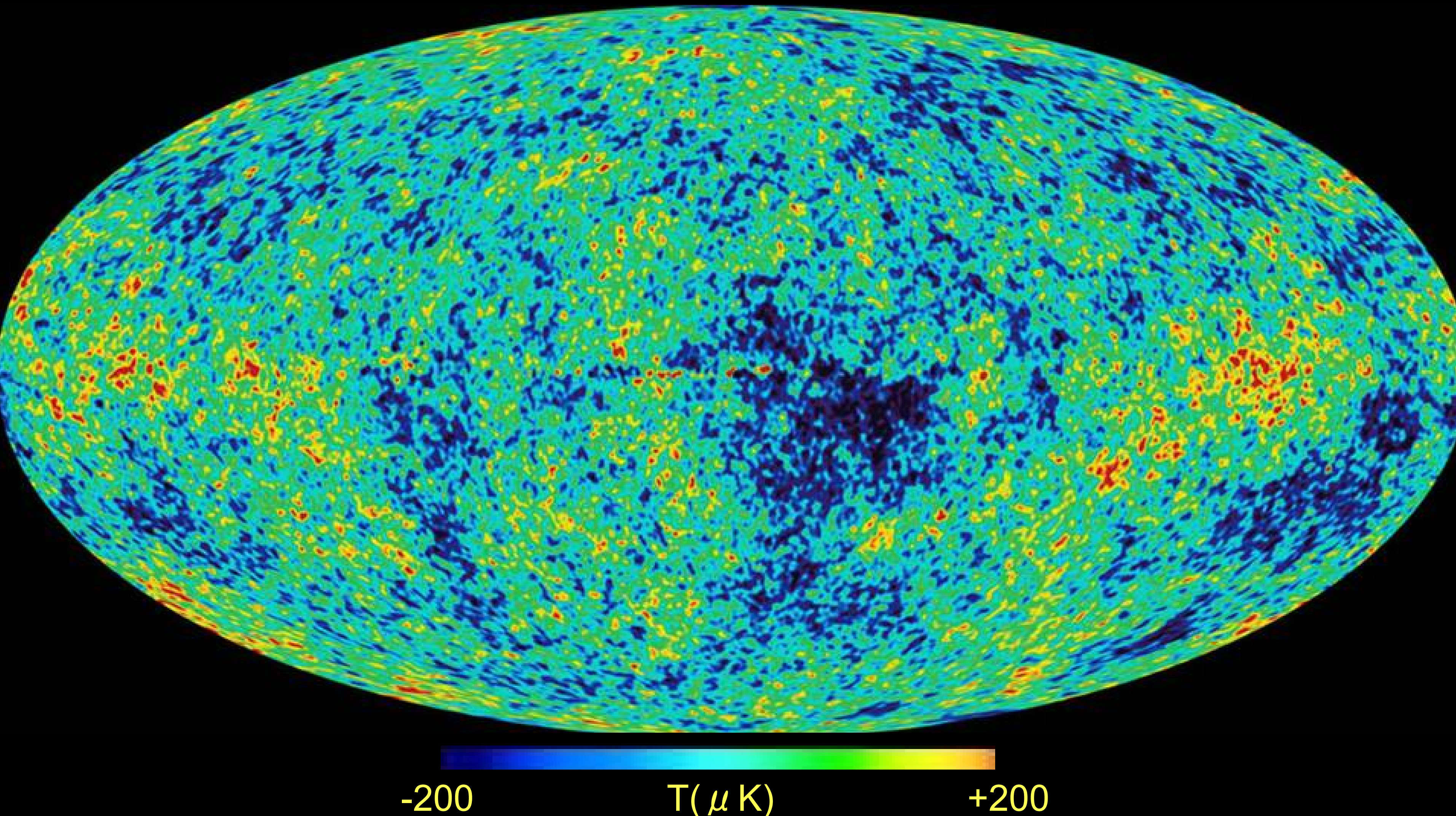
solves....

The Horizon Problem
The Flatness Problem
The monopole & other relic Problem
The origin-of-fluctuations Problem

Our Universe has hierarchical structures.

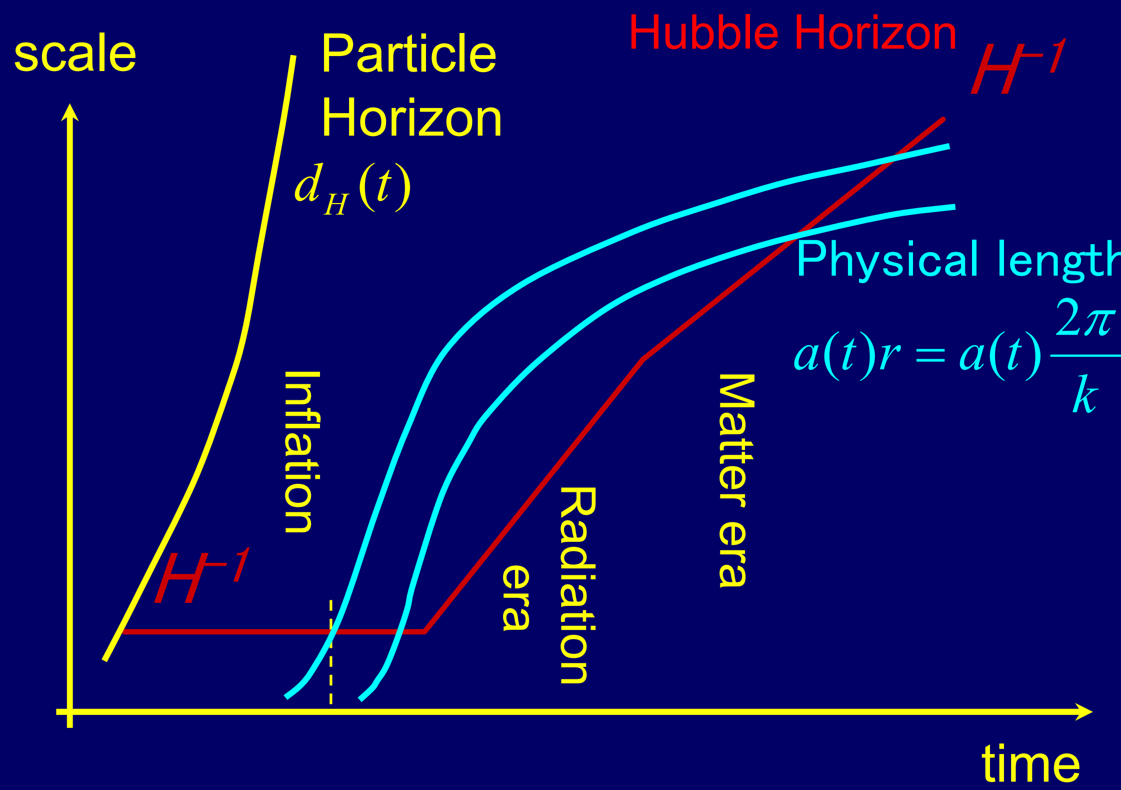


Their seed has also been observed as CMB anisotropy.



Temperature anisotropy at the level of 10^{-5}.

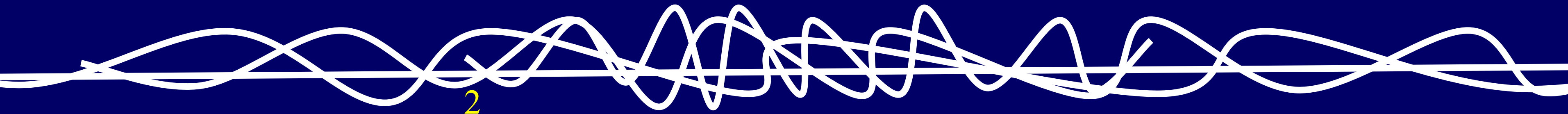
Evolution of scales in the Inflationary Cosmology



- ★ The particle horizon is exponentially stretched.
- ★ Each coordinate scale crosses the Hubble horizon twice, during and after inflation.
- ★ In between two horizon crossing epochs, that scale is beyond the Hubble radius and hence invisible.
- ★ During inflation, superhorizon fluctuations may be generated.

Inlation solves all these problems by

- Accelerated Expansion



- $a(t) \propto t^{\frac{2}{3(1+w)}}$ for $p = w\rho$

so $\boxed{w < -\frac{1}{3}}$ $\rho \propto a^{-3(1+w)}$ decreases less rapidly than the curvature term.

$$\boxed{w = -1 \Rightarrow a(t) \propto e^{Ht} \quad \rho = \text{const: } \Lambda_{\text{eff}}}$$

- Followed by Entropy Production

Reheating

Is inflation natural? Yes, if not always.

Cosmic No Hair Conjecture

If there exist a positive effective cosmological constant $\Lambda_{\rm eff}$(vacuum energy), then the Universe undergoes an exponential expansion within the Hubble time determined by the vacuum energy density.

It is easy to make counter examples, so it does not always hold.

Still there are proofs in some limited cases.

Homogeneous but anisotropic space (Wald 1983)

Bianchi type I～VIII (spatially flat or open), inflation occurs with $\Lambda_{\rm eff}$.

Bianchi type IX (positive curvature), inflation occurs if $\Lambda_{\rm eff} > \frac{1}{2} R^{(3)}_{\rm max}$.

maximum 3-curvature with fixed spatial volume

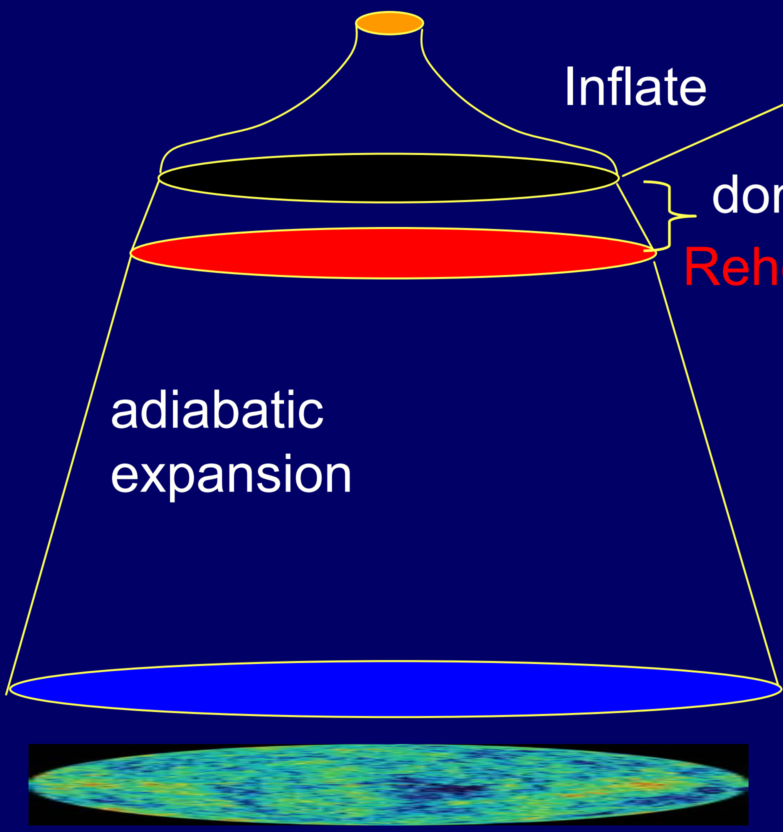
Inhomogeneous space

$\exists \Lambda_{\rm eff}$ Inflation occurs if $R^{(3)} < 0$ everywhere. (This condition is too strong.)

Numerical analysis suggests that if there exists $\Lambda_{\rm eff}$ and inhomogeneity in the corresponding Hubble volume is at most around unity, then inflation sets in for a wide class of initial conditions.

How much inflation is required to solve the horizon and the flatness problems?

The initial Hubble patch with radius H^{-1}



expand by $a_f / a_i \equiv e^N$ times

dominated by a field w/ EOS $P = w\rho$

Reheating

Reheating temperature T_R

The initial Hubble patch has expanded to

$$H^{-1} e^N \left(\frac{\pi^2 g_* T_R^4}{30 \rho_{\rm inf}} \right)^{-\frac{1}{3(1+w)}} \equiv r_H$$

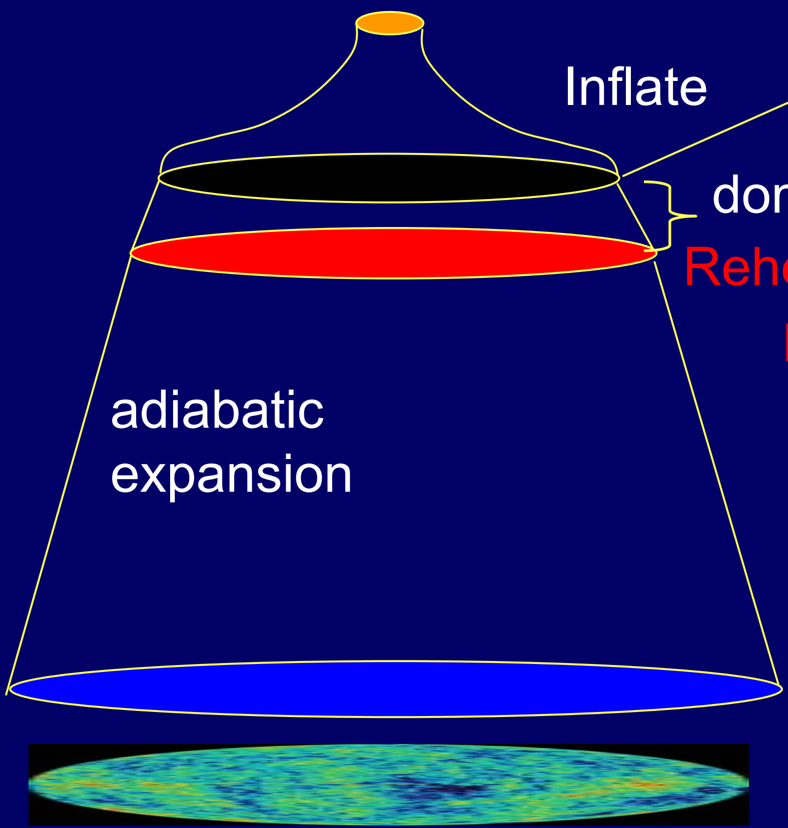
This region must be bigger than

the observable region, whose entropy is

given by $S_0 = 2.6 \times 10^{88}$ (2.7K CMB photon & 1.95K neutrinos × 3 generations in the Hubble radius $H_0^{-1} = 4.2 \times 10^3\,\mathrm{Mpc}$).

How much inflation is required to solve the horizon and the flatness problems?

The initial Hubble patch with radius H^{-1}



expand by $a_f / a_i \equiv e^N$ times

dominated by a field w/ EOS $P = w\rho$

Reheating

Reheating temperature T_R

The initial Hubble patch has expanded to

$$H^{-1}e^N\left(\frac{\pi^2 g_* T_R^4}{30\rho_{\rm inf}}\right)^{-\frac{1}{3(1+w)}} \equiv r_H$$

The entropy contained in this region is given by

$$S = \frac{4\pi^2 g_*}{90} T_R^3 \times \frac{4\pi}{3} r_H^3 = \frac{16\pi^3}{270}\left(\frac{45}{4\pi^3}\right)^{\frac{1}{1+w}} g_*^{\frac{w}{1+w}} \left(\frac{H}{M_{Pl}}\right)^{-\frac{1+3w}{1+w}} \left(\frac{T_R}{M_{Pl}}\right)^{\frac{-1+3w}{1+w}} e^{3N}$$

$$S=\frac{4\pi^2 g_*}{90}T_R^3\times\frac{4\pi}{3}r_H^3=\frac{16\pi^3}{270}\left(\frac{45}{4\pi^3}\right)^{\frac{1}{1+w}}g_*^{\frac{w}{1+w}}\left(\frac{H}{M_{Pl}}\right)^{-\frac{1+3w}{1+w}}\left(\frac{T_R}{M_{Pl}}\right)^{\frac{-1+3w}{1+w}}e^{3N}$$

must be larger than $S_0=2.6\times10^{88}$, so the number of e-folds N must satisfy

$$N>67.7-\frac{12.5-10.8w}{1+w}-\frac{w}{3+3w}\ln\left(\frac{g_*}{106.75}\right)+\frac{1+3w}{6(1+w)}\ln\left(\frac{r}{0.01}\right)$$
$$+\frac{1-3w}{3+3w}\ln\left(\frac{T_R}{10^8\mathrm{GeV}}\right)\equiv N_{\min},\quad r\equiv0.01\left(\frac{H}{2.4\times10^{13}\mathrm{GeV}}\right)^2$$

However, the above is merely a condition that the initial Hubble patch should have expanded larger than the current Hubble patch whose fluctuation is only at the level of 10^{-5}.

If the initial Hubble patch had fluctuations of order of unity, then it must expand by $\left(10^{-5}\right)^{-\frac{1}{2}}\sim500$ times longer than $N_{\min}$.

So the minimal condition for the number of e-folds reads

$N > N_{\min} + \ln 500 = N_{\min} + 6.2$, namely,

$$N > 55 + \frac{1}{6}\ln\left(\frac{r}{0.01}\right) + \frac{1}{3}\ln\left(\frac{T_R}{10^8\mathrm{GeV}}\right)$$

for $w = 0$. (standard inflation)

and

$$N > 67 - \frac{1}{6}\ln\left(\frac{g_*}{106.75}\right) + \frac{1}{3}\ln\left(\frac{r}{0.01}\right) - \frac{1}{3}\ln\left(\frac{T_R}{10^8\mathrm{GeV}}\right)$$

for $w = 1$. (k-inflation or G-inflation.)

Flatness of the Universe

$$\frac{K}{a^2} = H^2(\Omega_{\rm tot} - 1)$$

$$\frac{\Omega_{\rm tot}(t_0) - 1}{\Omega_{\rm tot}(t_i) - 1} = \left(\frac{a(t_i)H_{\rm inf}}{a_0 H_0}\right)^2 = e^{-2(N - N_{\rm min})} < 500^{-2} = 4 \times 10^{-6}$$

initial value at the onset of inflation

Prediction of Inflation I

If inflation solves the horizon problem, it predicts that our Universe is spatially flat with

$$\left|\Omega_{tot\,0} - 1\right| < 10^{-5}$$

Once inflation sets in, the Universe rapidly becomes homogeneous & isotropic, and almost spatially flat.

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

rapidly decreases

Anisotropic space

$$\bar{H}^2 + \frac{K(\ldots)}{\bar{a}^2} + \boxed{\frac{S(\ldots)}{\bar{a}^6}} = \frac{8\pi G}{3}\rho$$

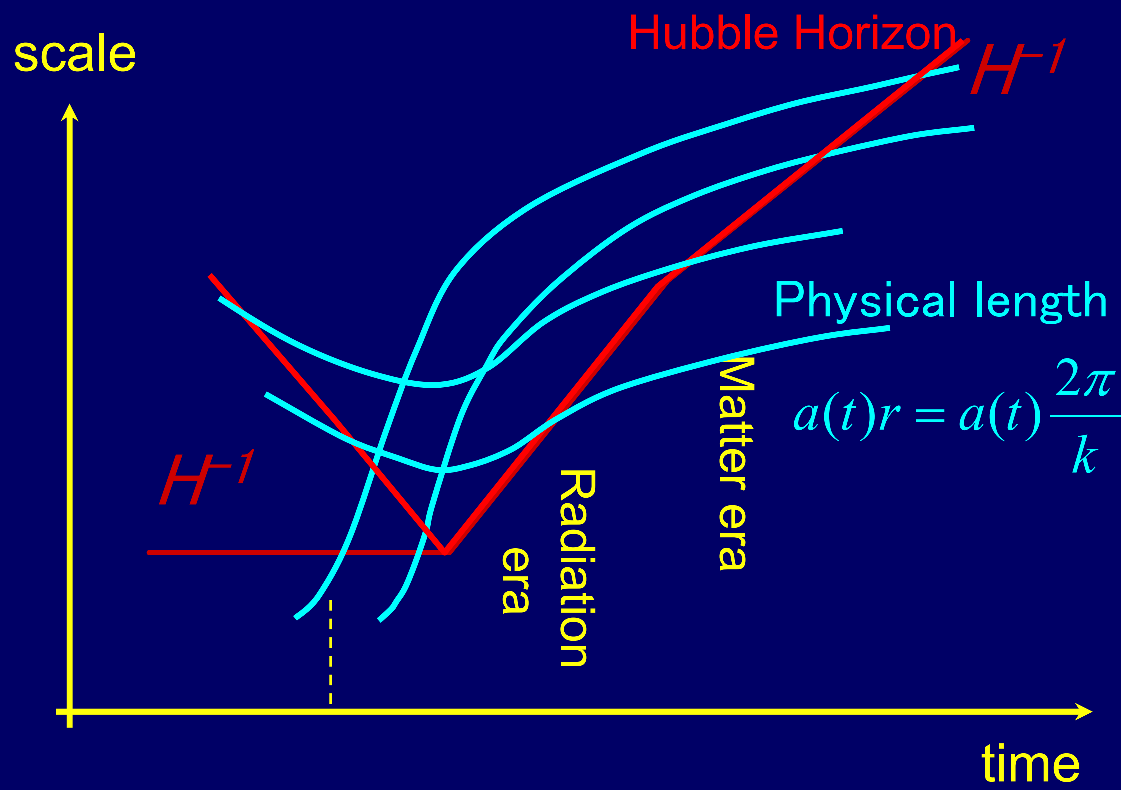
$$\bar{a} \equiv \sqrt[3]{\text{spatial volume factor}}$$

$$\bar{H} = \frac{\dot{\bar{a}}}{\bar{a}}$$

decreases with the same rate
as the spatial curvature
in the expanding phase.

Increases very rapidly
in a contraction phase
Problem for a
bouncing cosmology

Evolution of scales in the Bouncing Cosmology



-
- Each coordinate scale crosses the Hubble horizon twice, contraction and expansion stages.
- In between two horizon crossing epochs, that scale is beyond the Hubble radius and hence invisible.
- During bounce, superhorizon fluctuations may be generated.

上州無智亦無大
剛毅木訥易被欺
唯以正直接萬人
至誠信神期勝利
鑑

What drives INFLATION?

Energy density of a scalar field

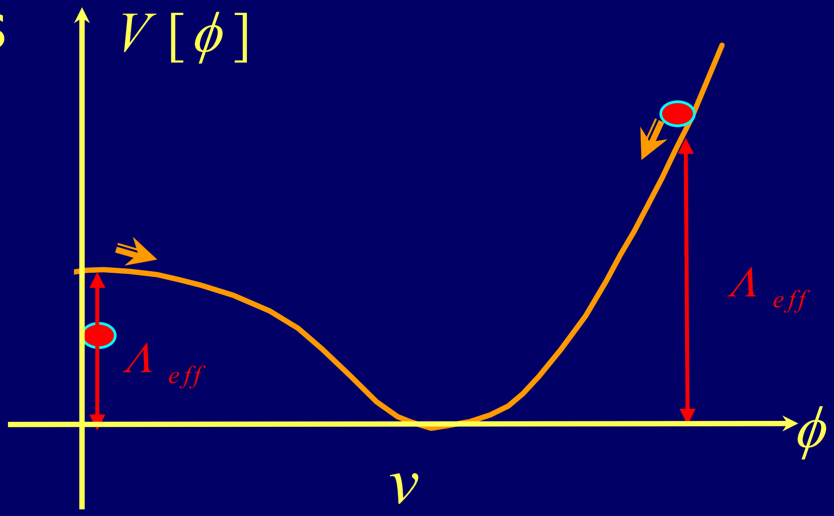
- A. Canonical Scalar Field

$$S=\int \mathcal{L}\sqrt{-g}\,d^4x=\int\left[-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V[\phi]\right]\sqrt{-g}\,d^4x$$

$$T_{\mu\nu}(x)=-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}(x)}=\partial_\mu\phi\partial_\nu\phi-\left[-\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-V[\phi]\right]g_{\mu\nu}$$

$$\boxed{\rho=\tfrac{1}{2}\dot{\phi}^2+V[\phi],\quad P=\tfrac{1}{2}\dot{\phi}^2-V[\phi]}$$

Inflation with $w=P/\rho=-1$ is realized if potential energy dominates.



Inflation driven by a canonical scalar field

$$\mathcal{L} = -\frac{1}{2} g_{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V[\phi] \quad\Rightarrow\quad \rho = \frac{1}{2}\dot{\phi}^2 + V[\phi], \quad P = \frac{1}{2}\dot{\phi}^2 - V[\phi]$$

Einstein equation

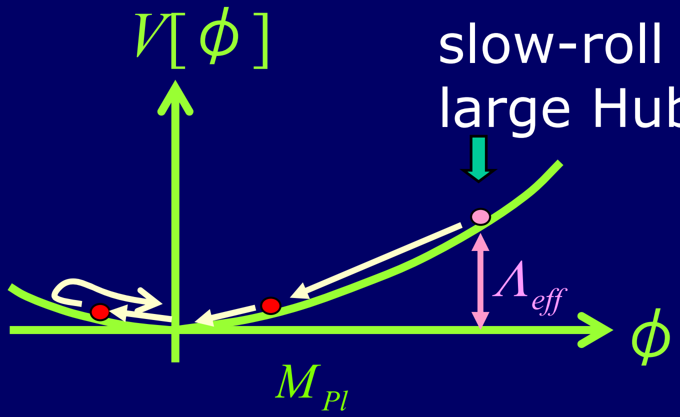
$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V[\phi]\right)$$

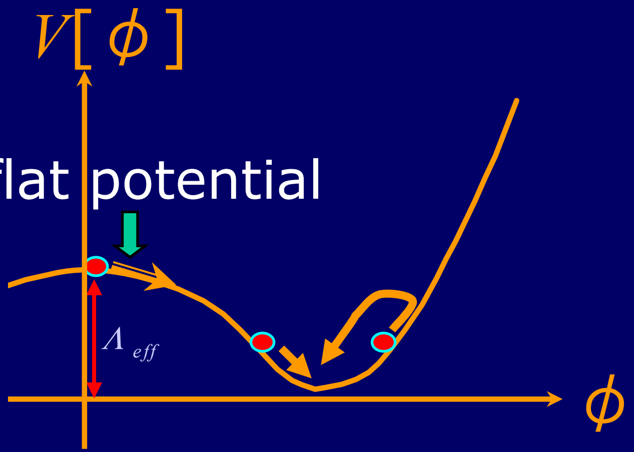
Field equation

$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0$$

$p = -\rho$

If energy density is dominated by the potential, inflation occurs.





Slow-roll equations of motion

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V[\phi]\right)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0$$

Inflation driven by a canonical scalar field

$$\mathcal{L} = -\frac{1}{2} g_{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V[\phi] \quad \Rightarrow \quad \rho = \frac{1}{2}\dot{\phi}^2 + V[\phi], \quad P = \frac{1}{2}\dot{\phi}^2 - V[\phi]$$

Einstein equation

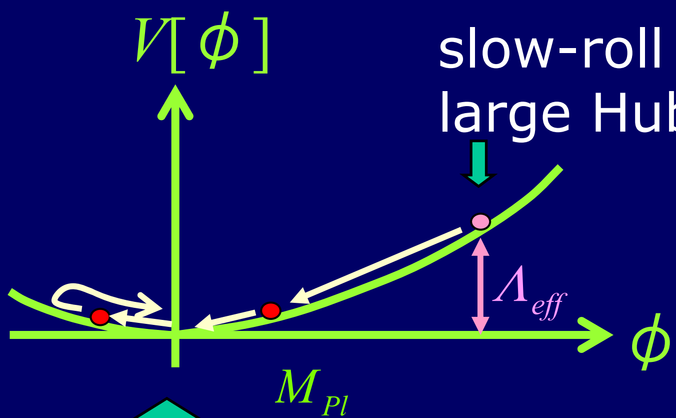
$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V[\phi]\right)$$

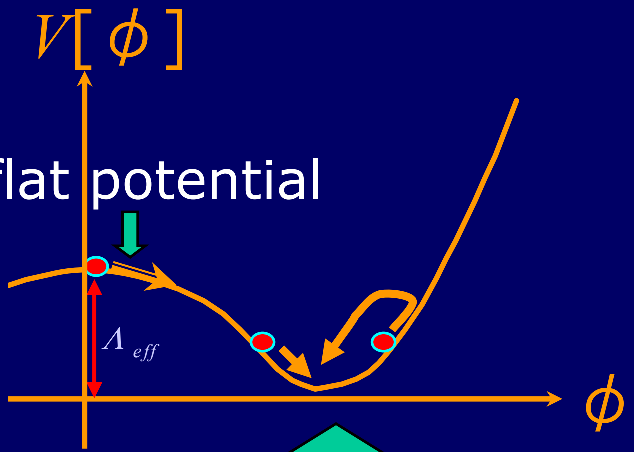
Field equation

$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0$$

$p = -\rho$

If energy density is dominated by the potential, inflation occurs.





field oscillation around the minimum
→particle production→reheating

But it is not a mandatory requirement…

What drives INFLATION?

Energy density of a scalar field

- B. Non-Canonical Scalar Field (k-inflation)

(Armendariz-Picon, Damour, & Mukhanov 99)

$$S=\int \mathcal{L}\sqrt{-g}\,d^4x=\int K(X,\phi)\sqrt{-g}\,d^4x,\quad X\equiv-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$T_{\mu\nu}(x)=Kg_{\mu\nu}-K_X(-\partial_\mu\phi\partial_\nu\phi)\equiv Pg_{\mu\nu}+(\rho+P)u_\mu u_\nu$$

$$\boxed{\rho=2XK_X-K,\quad P=K}\qquad K_X=\frac{\partial K}{\partial X}$$

Inflation is possible if $K<XK_X$.

Exponential inflation is realized if $K_X=0$.

For $K(X,\phi)=K_1(\phi)X+K_2(\phi)X^2$, $K_1(\phi)K_2(\phi)<0$ is required.

- C. with a Higher derivative term (G-inflation)

(Kobayashi, Yamaguchi, & JY 10)

$$\mathcal{L}_\phi=K(\phi,X)-G(\phi,X)\Box\phi\ldots,$$

The original references to Inflation

Scientific Background on the Nobel Prize in Physics 2011

THE ACCELERATING UNIVERSE

compiled by the Class for Physics of the Royal Swedish Academy of Sciences

In order to explain how the Universe can be so homogeneous with different parts that seemingly cannot have been in causal contact with each other, the idea of an inflationary phase in the early Universe was put forward [30].

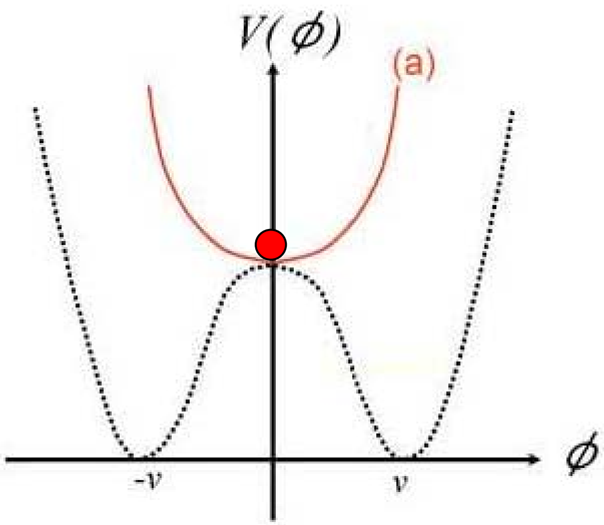
Old inflation (1st order phase transition)

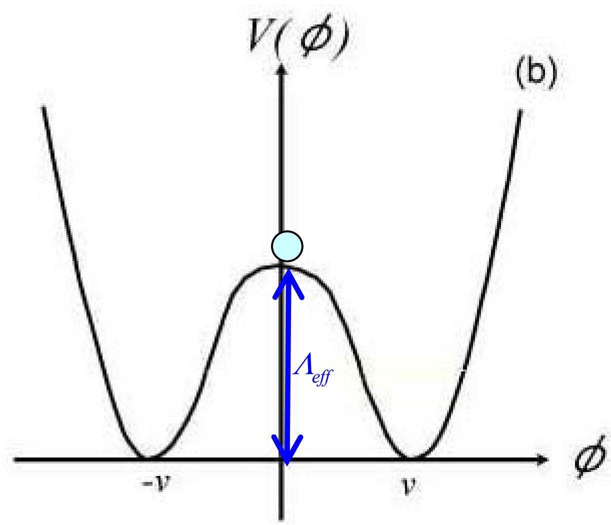
R^2 inflation (still viable)

[30] A. Starobinsky, "A new type of isotropic cosmological models without singularity", Phys. Lett., **B91**, 99-102, (1980);
K. Sato, "First order phase transition of a vacuum and expansion of the Universe", MNRAS, **195**, 467-479, (1981);
A.H. Guth, "The inflationary universe: A possible solution to the horizon and flatness problems", Phys. Rev., **D23**, 347-356, (1980);
A.D. Linde, "A new inflationary scenario: A possible solution to the horizon, flatness, homogeneity, isotropy and primordial monopole problems", Phys. Lett., **B108**, 389-393, (1981);
A. Albrecht and P.J. Steinhardt, "Cosmology for Grand Unified Theories with radiatively induced symmetry breaking", Phys. Rev. Lett., **48**, 1220-1223, (1982),

New inflation (slow-roll model)

Both old and new inflation models were based on the high-temperature symmetry restoration of grand unified theories in the early universe at around $T = 10^{15}$ GeV.





(a) $T >> v$

(b) $T = 0$

Was the early Universe in a thermal equilibrium state?

Two bodv reaction rate with a massless gauge particle

$$\Gamma_2 = \langle n\sigma c\rangle \simeq \frac{NT^3}{\pi^2}\frac{\alpha^2}{T^2}$$

N Number of reaction channel

α Gauge coupling constant

g_* # Relativistic degrees of freedom

must have been larger than

$$H = \left(\frac{8\pi}{3M_{PL}^2}\frac{\pi^2}{30}g_* T^4\right)^{\frac{1}{2}} .$$

Namely, $\Gamma \gg H$.

This imposes an upper bound on the radiation temperature,

$$T \ll 10^{15}\left(\frac{\alpha}{0.05}\right)^2\left(\frac{N}{10}\right)\left(\frac{g_*}{200}\right)^{-1/2}\mathrm{GeV} \equiv T_{\mathrm{eq}}$$

Thermal phase transition at the GUT scale was impossible.

Some nonthermal mechanisms to set up the initial condition for inflation must be invoked.

Example 1: Large-field model, Chaotic Inflation

(Linde 83)

★ Consider the simplest Lagrangian

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - V[\phi], \quad V[\phi] = \frac{1}{2}m^2\phi^2, \quad m \ll M_{Pl}$$

★ With a natural initial condition at the Planck epoch when the Universe was presumably dominated by large quantum fluctuations:

$$-\frac{1}{2}(\partial\phi)^2 \lesssim M_{Pl}^4, \qquad \frac{1}{2}m^2\phi^2 \lesssim M_{Pl}^4,$$

typical initial field amplitude

$$\phi \sim M_{Pl}^2/m \gg M_{Pl}$$

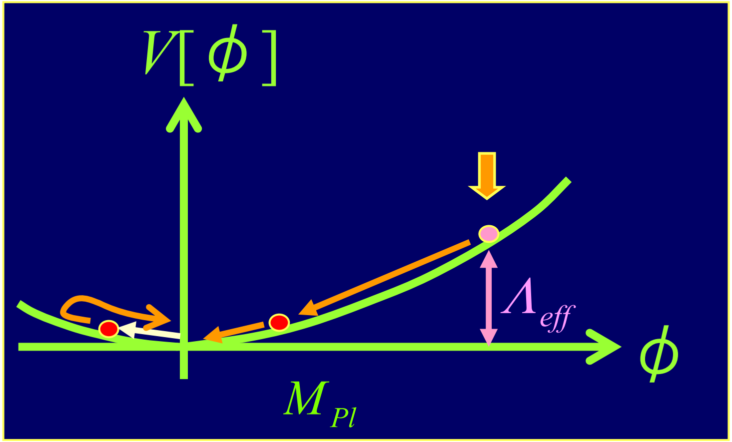
$$\frac{\phi^2}{2L^2} \leq M_{Pl}^4$$

typical size of coherent domain

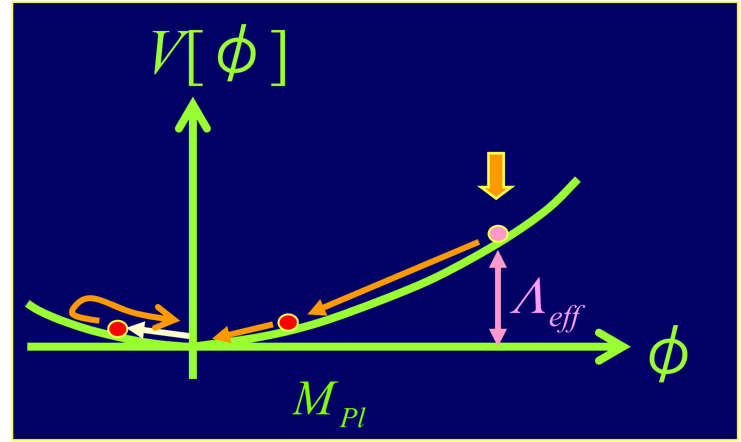
$$L \sim m^{-1} \gg M_{Pl}^{-1}$$ Horizon@ t_{Pl}

Compton wavelength of the field

"The Universe was smaller than a particle."



★This is a potential just for simple harmonic motion with a period $\tau = 2\pi/m$.
But when $\phi \gtrsim M_{Pl}$, we find $H \gtrsim m$
so the dynamics is friction dominated.



$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0,$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi\rho_\phi}{3M_{Pl}^2} = \frac{\rho_\phi}{3M_G^2} \qquad \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V[\phi], \qquad M_G = M_{Pl}/\sqrt{8\pi}$$

★ Slow-roll equations can be solved as

$$3H\dot{\phi} + V'[\phi] = 0, \qquad \left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi V[\phi]}{3M_{Pl}^2}$$

$$\phi(t) = \phi_i - \frac{mM_{Pl}}{2\sqrt{3\pi}}(t - t_i),$$
$$a(t) = a_i \exp\left[\sqrt{\frac{4\pi}{3}}\frac{m}{M_{Pl}}\phi_i(t - t_i) - \frac{m^2}{6}(t - t_i)^2\right]$$

★ Quasi-exponential inflation ends at $\phi \approx M_{Pl}/\sqrt{4\pi}$ when time variation rate $|\dot{\phi}/\phi|$ becomes as large as the cosmic expansion rate H .

★ After that the Universe is dominated by coherent field oscillation of ϕ .

☆ Number of e-folds

$$a(t) = a_i \exp\left[\frac{2\pi}{M_{Pl}^2}\left(\phi_i^2 - \phi^2(t)\right)\right] \Rightarrow N = \frac{2\pi}{M_{Pl}^2}\left(\phi_i^2 - \frac{M_{Pl}^2}{4\pi}\right)$$

$\phi_i \gtrsim 3M_{Pl}$ would be sufficient to solve the horizon problem.

☆ However, it is not trivial to have a flat enough potential for the field range beyond the Planck scale.
(Example) Supergravity inflation

$$D^i W \equiv \frac{\partial W}{\partial \phi_i} + \frac{3}{M_G^2}\frac{\partial K}{\partial \phi_i} W$$

$$V[\phi_i] = e^{\frac{K}{M_G^2}}\left[D^i W K_{i\bar{j}} D^{\bar{j}} W^* - \frac{3}{M_G^2}|W|^2\right] \qquad K_{i\bar{j}} = \left(\frac{\partial^2 K}{\partial \phi_i \partial \phi_{\bar{j}}^*}\right)^{-1} = \delta_{i\bar{j}}$$

Kahler potential also generates kinetic term.

$$\mathcal{L} = -K^{i\bar{j}} \partial\phi_i \partial\phi_{\bar{j}}^* - V[\phi_i] \qquad K^{i\bar{j}} \equiv \frac{\partial^2 K}{\partial \phi_i \partial \phi_{\bar{j}}^*}$$

For the minimal kinetic term $K^{i\bar{j}} = \delta^{i\bar{j}}$

$$K = \sum_i \phi_i \phi_i^* = \sum_i |\phi_i|^2 \Rightarrow$$

Exponentially steep potential beyond M_G.

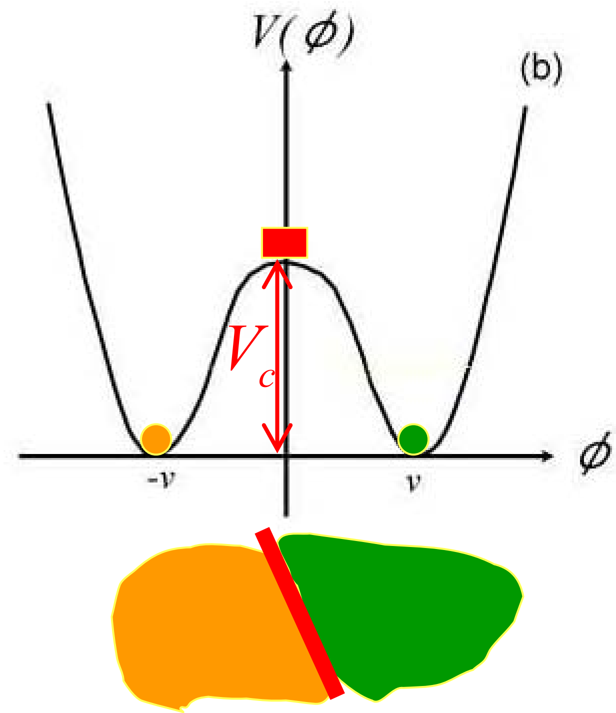
First successful model in SUGRA (Murayama, Suzuki, Yanagida, JY 93)
Shift symmetry (Kawasaki, Yanagida, Yamaguchi, 00)

☆ Stringy realization: Monodromy model (Silverstein & Westphal 08)

Example 2: Small-field model, Topological Inflation

(Vilenkin 94, Linde 94)

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V[\phi], \quad V[\phi] = \frac{\lambda}{4}(\phi^2 - v^2)^2$$



★ This model has a domain wall solution.
(Example) xy symmetric solution.

$$\phi(\boldsymbol{x}) = v\tanh\left(\sqrt{\frac{\lambda}{2}}vz\right)$$

★ Thickness of the wall is determined by the balance of $V[0] \equiv V_c$ and $(\partial\phi)^2 \sim \left(\frac{v}{d_0}\right)^2$ as

$$d_0 \approx vV_c^{-1/2} \approx \frac{1}{\sqrt{\lambda}v}$$

★ Comparing it with the Hubble radius corresponding to the energy density V_c

$$H_c^{-1} = \left(\frac{8\pi G}{3}V_c\right)^{-1/2} = M_{Pl}\left(\frac{3}{8\pi V_c}\right)^{1/2}$$

★ We find $d_0 \gtrsim H_c^{-1}$ if $v \gtrsim M_{Pl}$, that is, the domain wall is thicker than the Hubble horizon.

★ Inside the domain wall is dominated by a large potential energy $V \sim V_c$ of almost homogeneous field in the Hubble scale.

★ Such a region would inflate without respect to outside the domain wall.

Evolution of an inflating domain wall

★ Near the core of the wall, one can expand as $\phi(\boldsymbol{x}, t_c) \simeq kz$.

★ Since the spatial gradient is small here, one can solve the slow roll eqs at each point independently assuming $\mu^2 \equiv \lambda v^2 \ll H^2$ to yield

$$\phi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, t_c) \exp\left[\frac{\mu^2}{3H_c}(t - t_c)\right] = kz \exp\left[\frac{\mu^2}{3H_c}(t - t_c)\right]$$

$$a(t) \simeq a_c \exp[H_c(t - t_c)]$$

★ The coordinate, $z_*(t)$, where $\phi =^{\exists} \phi_*(\ll v)$ is given by

$$z_*(t) = k^{-1}\phi_* \exp\left[-\frac{\mu^2}{3H_c}(t - t_c)\right]$$

Any point with $z \neq 0$ will eventually reach $|\phi| > \phi_*$ and terminate inflation.

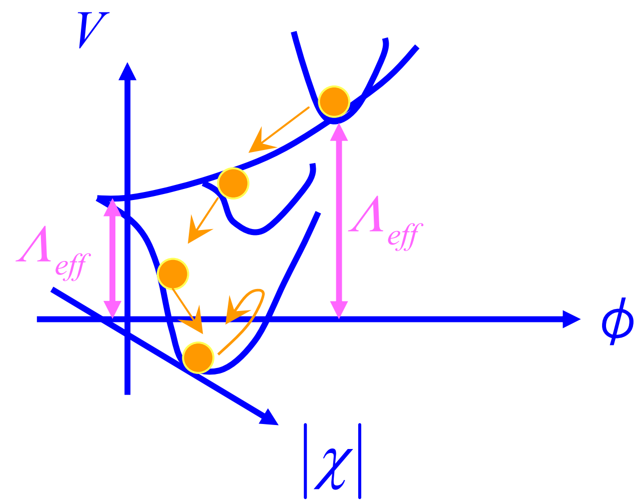
★ Its physical size will be exponentially stretched.

$$d(t) = a(t)z_*(t) = a_c k^{-1}\phi_* \exp\left[\left(H_c - \frac{\mu^2}{3H_c}\right)(t - t_c)\right]$$

Example 3: Vacuum dominated model, Hybrid Inflation

(Linde 94)

$$\mathcal{L} = -(\partial\chi)^\dagger(\partial\chi) - \frac{1}{2}(\partial\phi)^2 - V[\chi,\phi],$$



★ Symmetry restoration with another field

$$V[\chi,\phi] = \frac{\lambda}{2}(|\chi|^2 - v^2)^2 + g^2\phi^2|\chi|^2 + \frac{1}{2}m^2\phi^2$$

★ $g^2\phi^2 > \lambda v^2$: symmetry of χ is restored and false vacuum energy can drive inflation.

$$V[\chi = 0,\phi] = \frac{\lambda}{2}v^4 + \frac{1}{2}m^2\phi^2$$

$$\frac{\partial^2 V}{\partial\chi\partial\chi^\dagger} \equiv M_\chi^2 = \lambda(2|\chi|^2 - v^2) + g^2\phi^2$$

★ $\phi < \frac{\sqrt{\lambda}v}{g}$: phase transition and inflation ends.

★ Inflation can occur with field amplitudes much smaller than M_{Pl} .

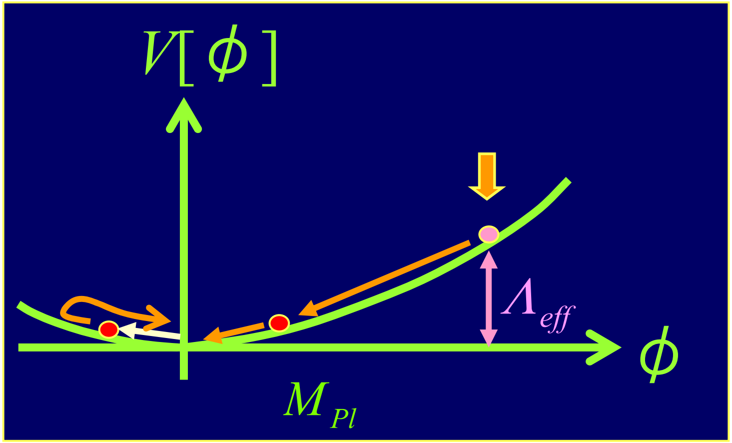
★ Fine tuning of initial condition of two fields is necessary.

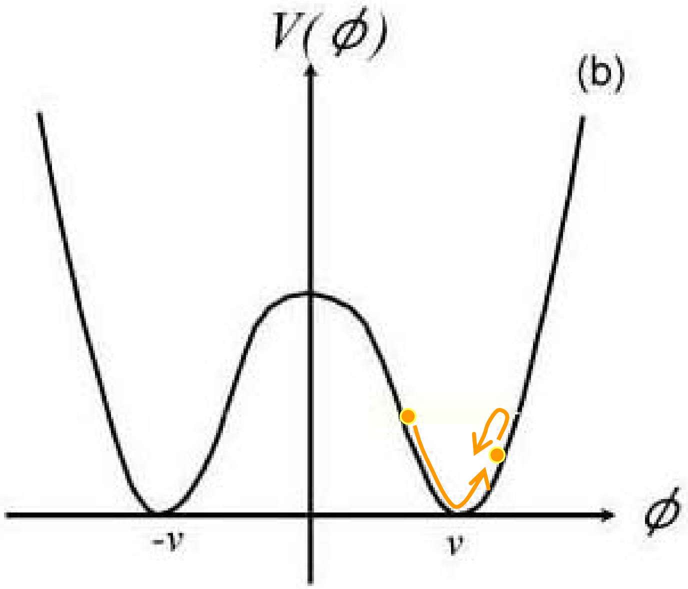
(Tetradis 98, Menders & Liddle 00)

After Inflation: Coherent Field Oscillation

- After potential-driven inflation, the scalar field oscillates around the global minimum.

- In some circumstances, *e.g.*, the case the minimum is at the origin and the inflaton is coupled to bosons, explosive particle production known as *preheating* takes place when the field amplitude is large.





- The coherent field oscillation is equivalent to the zero-mode condensate of the inflaton, and it decays with the decay rate of the inflaton particle eventually.
The final stage of reheating is governed by such a perturbative decay.

- For example, the Yukawa coupling $h\phi\bar{\psi}\psi$ gives the decay rate

$$\Gamma_\phi = \frac{h^2}{8\pi} m$$

★ When $H \gg \Gamma_\phi$, multiplying $\ddot{\phi}+3H\dot{\phi}+m^2\phi=0$ by $\dot{\phi}$ one finds

$$\frac{d}{dt}\underbrace{\left(\frac{1}{2}\dot{\phi}^2+\frac{1}{2}m^2\phi^2\right)}_{\rho_\phi} = -3H\dot{\phi}^2$$

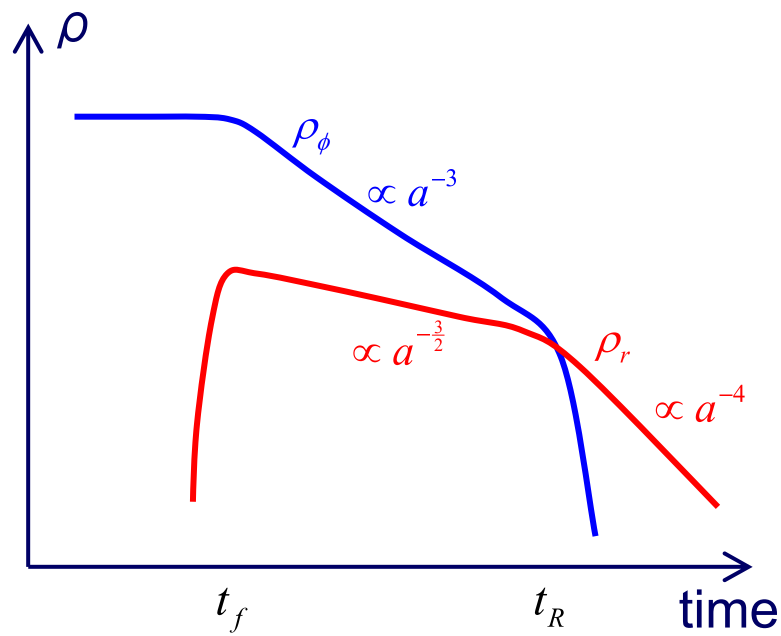
$\dot{\phi}^2$ can be replaced by $\overline{\dot{\phi}^2}$ due to rapid oscillation.

$$\rho_\phi = \overline{\dot{\phi}^2}$$

★ The energy transfer equations read

$$\frac{d\rho_\phi}{dt} = -(3H+\Gamma_\phi)\rho_\phi$$

$$\frac{d\rho_{\rm r}}{dt} = -4H\rho_{\rm r}+\Gamma_\phi\rho_\phi$$



★ The solutions are

$$\rho_\phi(t) = \rho_\phi(t_f)\left[\frac{a(t)}{a(t_f)}\right]^{-3}\exp[-\Gamma_\phi(t-t_f)] \qquad \rho_{\rm r}(t) = \Gamma_\phi\int_{t_f}^{t}\left[\frac{a(t)}{a(\tau)}\right]^{-4}\rho_\phi(\tau)d\tau$$

★ The Universe is dominated by radiation around $t_R \approx \Gamma_\phi^{-1}$
with the reheating temperature

$$T_R = 0.1 \left(\frac{200}{g_*}\right)^{1/4} \sqrt{M_{Pl}\Gamma_\phi} \cong 10^{11} \left(\frac{200}{g_*}\right)^{1/4} \left(\frac{\Gamma_\phi}{10^5\mathrm{GeV}}\right)^{1/2} \mathrm{GeV}$$

which is derived from an equality $H = \left(\frac{8\pi}{3M_{Pl}^2}\rho_r\right)^{1/2} = \left(\frac{8\pi}{3M_{Pl}^2}\frac{\pi^2 g_*}{30}T_R^4\right)^{1/2} \cong \frac{1}{2t} \cong \frac{1}{2}\Gamma_\phi$

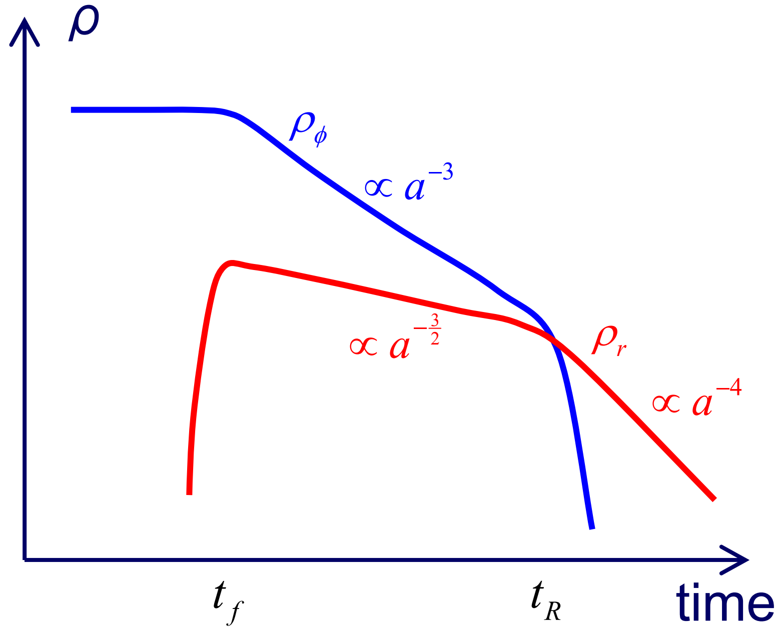
★ NB. The reheating temperature is not the maximum temperature after inflation but the temperature at the onset of radiation domination after significant entropy production.

$$\rho_r(t) = \Gamma_\phi \int_{t_f}^{t} \left[\frac{a(t)}{a(\tau)}\right]^{-4} \rho_\phi(\tau)d\tau$$

is solved as

$$\rho_r(t) = \frac{3}{5}\Gamma_\phi t\left[\frac{a(t)}{a(t_f)}\right]^{-3}\rho_\phi(t_f) \cong \frac{6}{5}\Gamma_\phi H M_G^2,$$

in the field oscillation regime.



★ The temperature decreases as

$$T \cong \left(\frac{36}{\pi^2 g_*}\Gamma_\phi H M_G^2\right)^{1/4} \searrow T_R$$

in the field oscillation regime, if the decay product is rapidly thermalized.

★ For the Yukawa coupling $h\phi\bar{\psi}\psi$ to a fermion with mass m_ψ,

the pertubative decay rate is

$$\Gamma_\phi = \frac{h^2}{8\pi} m_\phi \left[1 - \left(\frac{2m_\psi}{m_\phi}\right)^2\right]^{\frac{3}{2}} \equiv \Gamma_{\phi pert}$$

★ When the amplitude of oscillation is large, $h\phi_{amp} > m_\phi$, it is suppressed as

$$\Gamma_\phi = \frac{4\Gamma_{\phi pert}}{\pi^{5/2}} \left[\frac{m_\phi}{h\phi_{amp} \ln(h\phi_{amp}/m_\phi)}\right]^{\frac{1}{2}}$$

(Dolgov & Kirilova 90)

★ If the decay products are thermalized in the perturbative regime, the decay rate is modified as

$$\Gamma_\phi = \Gamma_{\phi pert}\left[1 - 2n_F\left(\frac{m_\phi}{2}\right)\right] \quad \text{to fermions}$$

or

$$\Gamma_\phi = \Gamma_{\phi pert}\left[1 + 2n_B\left(\frac{m_\phi}{2}\right)\right] \quad \text{to bosons}$$

After k-Inflation and G Inflation

ex $K(X,\phi) = K_1(\phi)X + K_2(\phi)X^2$

- Inflation ends when both coefficients turn to have positive sign.

- After inflation the Universe is dominated by the kinetic energy of ϕ , which now behaves as a free massless field,

$$\boxed{\rho = \frac{\dot{\phi}^2}{2} \propto a^{-6}(t).} \qquad w = 1$$

- Reheating occurs through gravitational particle production due to the change of the cosmic expansion law: $a(t) \propto e^{H_{\inf} t} \rightarrow a(t) \propto t^{\frac{1}{3}}$.

  At the end of inflation, radiation is created with its energy density corresponding at least to the Hawking temperature $T_H = H_{\inf}/2\pi$.

(Ford 87)

After k-Inflation and G Inflation

$$\frac{\rho_r}{\rho} \approx \frac{\frac{\pi^2}{30} g_* \left(\frac{H_{\inf}}{2\pi}\right)^4}{3M_{Pl}^2 H_{\inf}^2} \sim \left(\frac{H_{\inf}}{M_{Pl}}\right)^2 \ll 1$$ at the end of inflation.

$$\frac{\rho_r \propto a^{-4}(t)}{\rho \propto a^{-6}(t)} \nearrow \propto a^2(t)$$

- The Universe will eventually be dominated by radiation.

$$\boxed{T_R \approx 0.01 \frac{H_{\inf}^2}{M_{Pl}} = 2\times 10^7 \left(\frac{r}{0.1}\right) \mathrm{GeV}}$$

r : tensor-to-scalar ratio

- Massive particles with mass up to $\sim H_{\inf}$ are also copiously produced.

Baryogenesis through leptogenesis is possible if the mass of the lightest right-handed Majorana neutrino is smaller than the Hawking temperature.

上州無智亦無才
剛毅木訥易被欺
唯以正直接萬人
至誠依神期勝利
鑑三

Quantum properties of the inflaton

- ☆ Similar to the behavior of massless scalar field $\varphi(\boldsymbol{x},t)$ in de Sitter space whose square expectation value behaves as $\langle\varphi(\boldsymbol{x},t)^2\rangle = \left(\frac{H}{2\pi}\right)^2 Ht$.

(Bunchi & Davis 78, Vilenkin & Ford 82…)

- ☆ $$\varphi(\boldsymbol{x},t) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(\hat{a}_{\boldsymbol{k}}\varphi_k(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{\boldsymbol{k}}\varphi^*_k(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right) \equiv \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \hat{\varphi}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

- ☆ If we impose the normalization condition $\varphi_k(t)\dot{\varphi}^*_k(t) - \dot{\varphi}_k(t)\varphi^*_k(t) = \frac{i}{a^3(t)}$, the canonical commutation relation $[\varphi(\boldsymbol{x},t),\pi(\boldsymbol{x}',t)] = i\delta(\boldsymbol{x}-\boldsymbol{x}')$ yields $[\hat{a}_{\boldsymbol{k}},\hat{a}^\dagger_{\boldsymbol{k}'}] = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$ where the conjugate momentum is given by $\pi(\boldsymbol{x},t) = a^3(t)\dot{\varphi}(\boldsymbol{x},t)$.

- ☆ The mode function satisfies $$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3H\frac{\mathrm{d}}{\mathrm{d}t} + \frac{k^2}{e^{2Ht}}\right]\varphi_k(t) = 0$$ in de Sitter space and its normalized solution is given by

$$\varphi_k(t) = \sqrt{\frac{\pi}{4}}H(-\eta)^{3/2}H^{(1)}_{3/2}(-k\eta) = \frac{iH}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta}$$

$\eta \equiv \int^t \frac{dt}{a(t)} = \int^t \frac{dt}{e^{Ht}} = -\frac{1}{He^{Ht}}$ is the conformal time and $-k\eta = \frac{k}{Ha(t)}$.

$$\star\ \varphi_k = \frac{iH}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta} = \frac{iH}{\sqrt{2k^3}}\left(1-\frac{ik}{Ha}\right)e^{i\frac{k}{Ha}}$$

$$\to \frac{iH}{\sqrt{2k^3}}\left[1+O\left(\left(\frac{k}{Ha}\right)^2\right)\right], \quad \text{for } k \ll a(t)H$$

$\varphi_k^*(t) = -\varphi_k(t)$ in the superhorizon regime

So we find $\hat{\varphi}_{\boldsymbol{k}}(t) = \varphi_k(t)(\hat{a}_{\boldsymbol{k}} - \hat{a}^\dagger_{-\boldsymbol{k}})$

and its conjugate momentum reads

$$\hat{\pi}_{\boldsymbol{k}}(t) = a(t)^3\dot{\varphi}_k(t)(\hat{a}_{\boldsymbol{k}} - \hat{a}^\dagger_{-\boldsymbol{k}})$$

The same operator dependence!

$\star$ When the decaying mode is negligible, $\hat{\varphi}_{\boldsymbol{k}}$ and $\hat{\pi}_{\boldsymbol{k}}$ have the same operator dependence and commute with each other.

$\boxed{\text{Long-wave quantum fluctuations behave as if classical statistical fluctuations.}}$ ⇒ Origin of large scale structures and CMB anisotropy

★ In the short wave regime well inside the Hubble horizon, $k \gg aH$

$$\varphi_k = \frac{iH}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta} \longrightarrow \frac{-H\eta}{\sqrt{2k}}e^{-ik\eta} = \frac{1}{a\sqrt{2k}}e^{-ik\eta} = \frac{1}{a^{3/2}\sqrt{2k/a}}e^{-i\frac{k}{a}t} = \boxed{\frac{1}{a^{3/2}\sqrt{2k_{phys}}}e^{-ik_{phys}t}}$$

In a short time interval when cosmic expansion is negligible, we may set $dt = a(\eta)d\eta \longrightarrow t = a\eta$.

This is the usual positive frequency mode for the Minkowski vacuum with an unusual normalization

$$\varphi_k(t)\dot{\varphi}_k^*(t) - \dot{\varphi}_k(t)\varphi_k^*(t) = \frac{i}{a^3(t)}$$

$$\varphi_k(t) = \sqrt{\frac{\pi}{4}}H(-\eta)^{3/2}H_{3/2}^{(1)}(-k\eta)$$

defines the vacuum state with the appropriate Minkowski limit.

☆ The power spectrum reads

$$|\varphi_k(t)|^2 = \frac{H^2}{2k^3}(1+(k\eta)^2) \to \frac{H^2}{2k^3} \quad \text{for} \quad \frac{k}{Ha(t)} \longrightarrow 0$$

constant and proportional to k^{-3}

☆ Multiplying the phase space density, we find

$$|\varphi_k(t)|^2 \frac{4\pi k^3}{(2\pi)^3} d\ln k = \left(\frac{H}{2\pi}\right)^2$$

: scale-invariant fluctuation

$$\langle \varphi(\boldsymbol{x},t)^2 \rangle = \left(\frac{H}{2\pi}\right)^2 Ht$$

can be obtained by introducing IR and UV cutoffs as

$$\langle \varphi(\boldsymbol{x},t)^2 \rangle \simeq \int_H^{He^{Ht}} |\varphi_k(t)|^2 \frac{\mathrm{d}^3 k}{(2\pi)^3} = \left(\frac{H}{2\pi}\right)^2 Ht$$

: summing up superhorizon components generated during inflation

≈ Brownian motion with step $\pm\frac{H}{2\pi}$ and interval H^{-1}

In each Hubble time H^{-1}, quantum fluctuations with an amplitude $\delta\varphi \approx \pm\frac{H}{2\pi}$ and the initial wavelength $\lambda \approx H^{-1}$ is generated and stretched by inflation continuously.

★ For later convenience, we derive the same result starting from the action with the conformal time in the metric $ds^2 = a^2(\eta)(-d\eta^2 + d\boldsymbol{x}^2)$.

$$S = \int \sqrt{-g} d^4x \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right] = \frac{1}{2} \int d\eta d^3x \left\{ a^2 \left[\phi'^2 - (\nabla \phi)^2 \right] - a^4 m^2 \phi^2 \right\}$$

★ Using a rescaled field, $\chi \equiv a\phi$ the action is rewritten as

$$S = \frac{1}{2} \int d\eta d^3x \left[\chi'^2 - (\nabla \chi)^2 - \left(a^2 m^2 - \frac{a''}{a} \right) \chi^2 \right] \qquad \chi' = \frac{\partial \chi}{\partial \eta}$$

after integration by parts. So it is of the same form as a free-scalar action with a time dependent mass.

★ In the de Sitter background, $a(\eta) = -1/(H\eta)$, the mode function χ_k satisfies

$$\chi_k'' + k^2 \chi_k - \frac{2}{(-\eta)^2} \chi_k = 0$$

★ The solution satisfying the normalization condition $\chi' \chi^* - \chi \chi^{*'} = i$ as in the Minkowski space is given by

$$\chi_k(\eta) = \left(-\frac{\pi \eta}{4} \right)^{1/2} H_{3/2}^{(1)}(-k\eta) = \frac{\varphi_k(t)}{a(t)}$$

in agreement with the previous calculation.

Cosmological perturbation theory

★ Incorporate linear perturbation to the FLRW background $\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \mathrm{d}\boldsymbol{x}^2$.

$$\mathrm{d}s^2 = -(1+2A)\mathrm{d}t^2 - 2aB_j \mathrm{d}t\mathrm{d}x^j + a^2(\delta_{ij} + 2H_L\delta_{ij} + 2\underline{H_{T\,ij}})\mathrm{d}x^i\mathrm{d}x^j$$

traceless $\quad i, j = 1, 2, 3$

★ Decompose perturbation variables to spatial scalar, vector, and tensor.

$$B_j = \partial_j B + \widetilde{B}_j, \quad \partial_j \widetilde{B}_j = 0$$ (rotation free mode + divergence free mode)

$$H_{T\,ij} = \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\nabla^2\right) H_T + \partial_i \widetilde{H}_{T\,j} + \partial_j \widetilde{H}_{T\,i} + H_{TT\,ij}$$

transverse-traceless mode

$$\partial_j \widetilde{H}_{T\,j} = 0, \quad \partial_j H_{TT}{}^k{}_j = 0, \quad H_{TT}{}^j{}_j = 0$$

$A,\ B,\ H_L\ \&\ H_T$ Scalar modes･･･Density/Curvature Fluctuations

$\widetilde{B}_j\ \&\ \widetilde{H}_{T\,j}$ Vector modes･･･Decaying modes only

$H_{TT\,ij}$ Tensor modes･･･Gravitational Waves

★ In the linear perturbation theory, scalar, vector, and tensor modes are decoupled from each other. Each Fourier mode also behaves independently.

★ First consider scalar modes in Fourier space

$$ds^2 = -(1+2AY)dt^2 - 2aBY_j dt dx^j + a^2(\delta_{ij} + 2H_L Y \delta_{ij} + 2H_T Y_{ij}) dx^i dx^j$$

Y, Y_i, Y_{ij} are scalar harmonics defined by

$$Y = Y_{\boldsymbol{k}} \equiv e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad Y_i \equiv -i\frac{k_i}{k}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad Y_{ij} \equiv \left(-\frac{k_i k_j}{k^2} + \frac{1}{3}\delta_{ij}\right) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

★ Here "AY" means $AY = \sum_{\boldsymbol{k}} A_{\boldsymbol{k}} Y_{\boldsymbol{k}} = \int \frac{d^3k}{(2\pi)^3} A_{\boldsymbol{k}} Y_{\boldsymbol{k}}$ etc.

★ Each perturbation variable is a quantity in Fourier space, e.g. $A = A_{\boldsymbol{k}}(t)$.

★ Physical meaning of each perturbation variable.

A: Fluctuation of the lapse function (Newtonian Potential)

B: Fluctuation of the shift vector

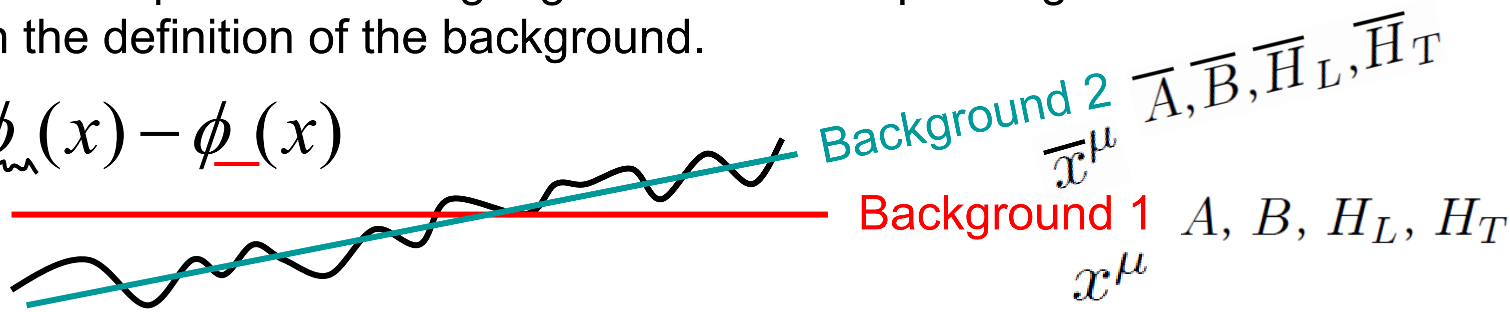
H_L: Fluctuation of the spatial volume

H_T: Spatial anisotropy

Issues of the Gauge

- Here we started from the background FLRW spacetime and then incorporated perturbations. But actually the real entity is an inhomogeneous spacetime which may be decomposed to a background and perturbations around it. The definition of the background is not unique. We have gauge modes corresponding to the freedoms associated with the definition of the background.

$$\delta\phi(x) = \phi_{\sim}(x) - \phi_{-}(x)$$



- To see how the gauge modes appear, we introduce two coordinate systems corresponding to Background 1 (x^μ) and 2 ($\overline{x}^\mu$) and compare expressions of perturbation variables at the same coordinate value.

- Suppose that two coordinates are related by the following scalar-type transformation.

$$\overline{x}^0 = x^0 + \delta x^0 = x^0 + TY \qquad \overline{x}^i = x^i + \delta x^i = x^i + \underline{LY^i}$$

gradient of a scalar

- Then the metrices of the two coordinates are related as

$$\begin{aligned}\overline{g}_{\mu\nu}(x) &= \frac{\partial x^\alpha}{\partial \overline{x}^\mu}\frac{\partial x^\beta}{\partial \overline{x}^\nu} g_{\alpha\beta}(x - \delta x) \\ &= g_{\mu\nu}(x) - g_{\alpha\nu}(x)(\delta x^\alpha)_{,\mu} - g_{\mu\beta}(x)(\delta x^\beta)_{,\nu} - g_{\mu\nu,\lambda}(x)\delta x^\lambda\end{aligned}$$

★ In terms of perturbation variables we find

$$\overline{A} = A - \dot{T}, \qquad \overline{H}_L = H_L - \frac{k}{3}L - HT,$$

$$\overline{B} = B + a\dot{L} + \frac{k}{a}T, \qquad \overline{H}_T = H_T + kL,$$

★ We can constitute two functions independent of generators L and T , namely, gauge invariant quantities.

$$\Phi_A \equiv A + \frac{a}{k}\dot{B} + \frac{\dot{a}}{k}B - \frac{a^2}{k^2}\left(\ddot{H}_T + 2\frac{\dot{a}}{a}\dot{H}_T\right) = \Psi = \Phi$$

$$\Phi_H \equiv H_L + \frac{1}{3}H_T + \frac{\dot{a}}{k}B - \frac{a\dot{a}}{k^2}\dot{H}_T = \Phi = -\Psi$$

$$\mathcal{R} \equiv H_L + \frac{1}{3}H_T$$

curvature perturbation

Japanese notation

Kodama&Sasaki PTP Suppl 78(1984)1

★ Gauge-invariant variables can be defined similarly for matter contents, too.

(Example) A scalar field transforms as $\phi(x) = \bar{\phi}(\bar{x})$ by definition.

$$\phi(t, \boldsymbol{x}) = \phi(t) + \Delta\phi Y$$

$$\downarrow$$

$$\begin{aligned}\bar{\phi}(t, \boldsymbol{x}) &= \phi(t - TY, x^j - LY^j) = \phi(t - TY) + \Delta\phi Y \\ &= \phi(t) - \dot{\phi}(t) TY + \Delta\phi Y\end{aligned}$$

$$\therefore \ \overline{\Delta\phi} = \Delta\phi - \dot{\phi} T$$

$$\left\{\begin{aligned}\bar{B} &= B + a\dot{L} + \frac{k}{a}T, \\ \bar{H}_T &= H_T + kL,\end{aligned}\right.$$

$$\delta\phi = \Delta\phi + \frac{a}{k}\left(B - \frac{a}{k}\dot{H}_T\right)\dot{\phi}$$

gauge-invariant scalar field perturbation

Fixing the gauge

- In fact, we do not need to start with the most general metric and consider gauge transformation to find invariant quantities, but it is sufficient if the gauge degrees of freedom, L and T are fixed.

$$\overline{A} = A - \dot{T}, \qquad \overline{H}_L = H_L - \frac{k}{3}L - HT,$$

$$\overline{B} = B + a\dot{L} + \frac{k}{a}T, \qquad \overline{H}_T = H_T + kL, \qquad \overline{\Delta\phi} = \Delta\phi - \dot{\phi}T$$

- Longitudinal Gauge

Let $H_T \equiv 0$ then L is fixed. Then let $B \equiv 0$, then T is also fixed.

$$A \equiv \Phi_A, \; H_L \equiv \Phi_H \Leftarrow \begin{aligned} \Phi_A &\equiv A + \frac{a}{k}\dot{B} + \frac{\dot{a}}{k}B - \frac{a^2}{k^2}\left(\ddot{H}_T + 2\frac{\dot{a}}{a}\dot{H}_T\right) \\ \Phi_H &\equiv H_L + \frac{1}{3}H_T + \frac{\dot{a}}{k}B - \frac{a\dot{a}}{k^2}\dot{H}_T \,. \end{aligned}$$

$$\mathrm{d}s^2 = -(1 + 2\Phi_A Y)\mathrm{d}t^2 + a^2(1 + 2\Phi_H Y)\mathrm{d}\boldsymbol{x}^2$$

$$\delta\phi = \Delta\phi$$

Fixing the gauge

- In fact, we do not need to start with the most general metric and consider gauge transformation to find invariant quantities, but it is sufficient if the gauge degrees of freedom, L and T are fixed.

$$\overline{A} = A - \dot{T}, \qquad \overline{H}_L = H_L - \frac{k}{3}L - HT,$$

$$\overline{B} = B + a\dot{L} + \frac{k}{a}T, \qquad \overline{H}_T = H_T + kL, \qquad \overline{\Delta\phi} = \Delta\phi - \dot{\phi}T$$

- **Unitary Gauge**

Let $\overline{\Delta\phi} = \Delta\phi - \dot{\phi}T = 0$, then T is fixed.

Also let $H_T \equiv 0$, then L is fixed, too.

$\Delta\phi = 0$: scalar field is homogeneous.

Evolution equations for perturbations

☆ For the moment, we work in the longitudinal gauge

$$\mathrm{d}s^2 = -(1+2\Phi_A Y)\mathrm{d}t^2 + a^2(1+2\Phi_H Y)\mathrm{d}\boldsymbol{x}^2$$

and introduce scalar-type perturbations to the perfect fluid matter.

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho+P)u^\mu u^\nu \quad (u^\mu u_\mu = -1)$$

$$\rho \to \rho + \delta\rho Y, \qquad u^\mu = (1,0,0,0) \to (1-AY, vY^j/a),$$

$$P \to P + \delta P Y, \qquad u_\mu = (-1,0,0,0) \to (-1-AY, avY_j)$$

Since the gauge is already fixed, these variables are also gauge-invariant.

☆ Write down the perturbed Einstein equations $\delta G^\mu{}_\nu = 8\pi G \delta T^\mu{}_\nu$

Hamiltonian constraint

$$\delta T^0{}_0 = -\rho\delta Y$$

$$\delta T^j{}_0 = -(\rho+P)\frac{v}{a}Y^j$$

$$\delta T^0{}_j = a(\rho+P)vY_j$$

$$\delta T^i{}_j = P(\pi_L \delta^i_j Y + \pi_T Y^i{}_j)$$

$$\delta \equiv \delta\rho/\rho$$

$$\pi_L \equiv \delta P/P$$

$$\pi_T = 0$$

anisotropic stress

$$\delta G^0{}_0 = \left(6H^2\Phi_A - 6H\dot{\Phi}_H - 2\frac{k^2}{a^2}\Phi_H\right)Y$$

$$\delta G^j{}_0 = \left(-2\frac{kH}{a^2}\Phi_A + 2\frac{k}{a^2}\dot{\Phi}_H\right)Y^j$$

$$\delta G^0{}_j = (2kH\Phi_A - 2k\dot{\Phi}_H)Y_j$$

Dynamical eq.

$$\delta G^i{}_j = \left[\left(2H^2 + 4\frac{\ddot{a}}{a}\right)\Phi_A + 2H\dot{\Phi}_A - \frac{2}{3}\frac{k^2}{a^2}\Phi_A - 6H\dot{\Phi}_H - 2\ddot{\Phi}_H - \frac{2}{3}\frac{k^2}{a^2}\Phi_H\right]\delta^i_j Y - \frac{k^2}{a^2}(\Phi_A+\Phi_H)Y^i{}_j$$

★ From Hamiltonian and momentum constraints we find

$$2\frac{k^2}{a^2}\Phi_H = 8\pi G\rho\Delta = 3H^2\Delta \qquad \Delta \equiv \delta + 3(1+w)\frac{aH}{k}v \quad (w \equiv P/\rho)$$

is the comoving density perturbation.

★ $Y^i{}_j$ term yields $\Phi_H + \Phi_A = 0$ ①

★ As a result we find the Poisson equation $-\frac{k^2}{a^2}\Phi_A = 4\pi G\rho\Delta$ ②

★ Dynamical equation may be found from $\delta^i_j Y$ term or from $\delta T^\mu{}_{\nu;\mu} = 0$.

$$\dot{\Delta} - 3Hw\Delta = -(1+w)\frac{k}{a}v$$ ③ continuity eqn.

$$\dot{v} + Hv = \frac{1}{\rho+P}\frac{k}{a}(\delta P - {c_s}^2\delta\rho + {c_s}^2\rho\Delta) + \frac{k}{a}\Phi_A$$ ④ Euler eqn. $c_s^2 \equiv \frac{dP}{d\rho} = \frac{\dot{P}}{\dot{\rho}}$

★ From ①②③④, we find

$$\dot{\Phi}_H + H\Phi_H = -4\pi G(\rho+P)\frac{a}{k}v \equiv -\frac{3}{2}(1+w)H\Upsilon \qquad \Upsilon \equiv \frac{aH}{k}v$$

Momentum constraint

$$\dot{\Upsilon} + \frac{3}{2}H(1+w)\Upsilon = -H\Phi_H + \frac{H}{1+w}({c_s}^2\Delta + w\Gamma)$$

Euler eqn.

$$p\Gamma \equiv \delta P - {c_s}^2\delta\rho$$

★ Subtracting each other we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_H - \Upsilon) = -\frac{H}{1+w}({c_s}^2\Delta + w\Gamma) \qquad 2\frac{k^2}{a^2}\Phi_H = 8\pi G\rho\Delta = 3H^2\Delta$$

If there are only adiabatic fluctuations, e.g., single fluctuating component, we find $p\Gamma \equiv \delta P - {c_s}^2\delta\rho = 0$, since $c_s^2 = \dot{P}/\dot{\rho} = \delta P/\delta\rho$ holds. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_H - \Upsilon) = -\frac{H}{1+w}{c_s}^2\Delta = -\frac{2{c_s}^2 H}{3(1+w)}\left(\frac{k}{aH}\right)^2\Phi_H \quad 0 \text{ for } \frac{k}{aH} \to 0$$

superhorizon limit

★ Comoving curvature perturbation

$$\Phi_H - \Upsilon = \mathcal{R} - \frac{aH}{k}v \equiv \mathcal{R}_c$$

is conserved outside the Hubble radius if only adiabatic fluctuations are present.

★ In the case of single scalar-field matter with $\mathcal{L} = K(X,\phi)$, we find

$${c_s}^2\Delta + w\Gamma = \tilde{c}_s^2\Delta \quad \text{with} \quad \tilde{c}_s^2 \equiv \frac{P_X}{\rho_X} = \frac{K_X}{K_X + 2XK_{XX}}$$

so the conservation of comoving curvature perturbation also holds.

This gives the sound velocity of a scalar field.
It is unity for canonical fields.

★ Using the momentum constraint we can express $\mathcal{R}_c$ with Φ_H only.

$$\mathcal{R}_c = \Phi_H - \Upsilon = \Phi_H + \frac{2}{3(1+w)}(\Phi_H + H^{-1}\dot{\Phi}_H) \equiv \zeta$$ Bardeen's ξ

★ $\zeta = \mathrm{const} \equiv C_1$ can be solved as a first-order differential equation

★ The solution is given by $\Phi_H = C_1\left(1 - \frac{H}{a}\int^t a(t')\mathrm{d}t'\right)$

namely,

Growing adiabatic mode $\Phi_H^G = C_1\left(1 - \frac{H}{a}\int^t a(t')\mathrm{d}t'\right)$

Decaying adiabatic mode $\Phi_H^D = \frac{H}{a}$

★ When $w = P/\rho = \mathrm{const}$ we find $\Phi_H^G = C_1\left(1 + \frac{2}{3(1+w)}\right)^{-1} = \begin{cases} \frac{2}{3}C_1 & \text{for } w = \frac{1}{3} \\ \frac{3}{5}C_1 & \text{for } w = 0 \end{cases}$

★ In a contracting phase the “Decaying mode” $\Phi_H^D = \frac{H}{a}$ grows severely.

上州人

上州無智亦無才
剛毅木訥易被欺
唯以正直接萬人
至誠依神期勝利

鑑三

Curvature perturbation from inflation

★ Incorporate curvature perturbation to FLRW Universe and calculate its action in the Einstein+scalar model.

$$S = \int d^4x\sqrt{-g}\left[\frac{M_G^2}{2}R + K(X,\phi)\right]$$

include both potential-driven and k-inflation models

★ Background equations

$$3M_G^2H^2 = \rho = 2XK_X - K, \qquad 2M_G^2\dot{H} + 3M_G^2H^2 = P = K,$$

$$\ddot{\phi} + 3Hc_s^2\dot{\phi} + \frac{K_{X\phi}}{K_X}c_s^2\dot{\phi}^2 - \frac{K_\phi}{K_X}c_s^2 = 0, \quad \text{with} \quad c_s^2 \equiv \frac{P_X}{\rho_X} = \frac{K_\phi}{K_X + 2XK_{XX}}$$

sound speed of perturbation

★ We adopt 3+1 ADM decomposition which is useful to separate constraint equations.

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^idt)(dx^j + N^jdt), \qquad h_{ij} \equiv a^2(t)e^{2\mathcal{R}}\delta_{ij}$$

$\mathcal{R} \equiv \mathcal{R}_c$ represents comoving curvature perturbation conserved outside the horizon.

No gauge mode in L, since we have $\overline{H}_T = H_T + kL = 0$.

Setting $\overline{\Delta\phi} = \Delta\phi - \dot{\phi}T = 0$, gauge in T is also fixed.

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad h_{ij} \equiv a^2(t) e^{2\mathcal{R}} \delta_{ij}$$

★ The action then reads

$$S = \frac{1}{2}\int d^4x \sqrt{h} N (M_G^2 R^{(3)} + 2K) + \frac{M_G^2}{2}\int d^4x \sqrt{h} N^{-1}(E_{ij}E^{ij} - E^2) +$$

where $E_{ij} \equiv \frac{1}{2}(\dot{h}_{ij} - N_{i|j} - N_{j|i}), \quad E \equiv \mathrm{Tr} E$

Total derivative terms not affecting field eqs

★ We set $N = 1 + \alpha, \; N_i = \partial_i \psi$ to analyze linear scalar perturbations.

perturbation variables

★ The Hamiltonian constraint obtained by differentiation w.r.t. N reads

$$R^{(3)} + 2\frac{K}{M_G^2} - 4X\frac{K_X}{M_G^2} - \frac{1}{N^2}(E_{ij}E^{ij} - E^2) = 0$$

$$\frac{H}{a^2}\partial^2\psi = -\frac{1}{a^2}\partial^2\mathcal{R} + \Sigma\alpha, \quad M_G^2 \Sigma \equiv XK_X + 2X^2 K_{XX}$$

★ The momentum constraint obtained by differentiation w.r.t. N^i reads

$$\left[\frac{1}{N}(E_i^j - E\delta_i^j)\right]_{|j} = 2H\alpha_{,i} - 2\dot{\mathcal{R}}_{,i} = 0 \quad \Rightarrow \quad \alpha = \dot{\mathcal{R}}/H$$

★ Now that both α and ψ have been expressed by $\mathcal{R}$, we can obtain the second order action for $\mathcal{R}$ as

$$S_2 = M_G^2 \int dt d^3x a^3 \left[\frac{\Sigma}{H^2}\dot{\mathcal{R}}^2 - \varepsilon_H \frac{(\partial\mathcal{R})^2}{a^2}\right], \quad \varepsilon_H \equiv -\frac{\dot{H}}{H^2}$$

★ Introducing new variables, $z \equiv \frac{a\sqrt{2\Sigma}}{H} = \frac{a\sqrt{2\varepsilon_H}}{c_s}$, and $v \equiv M_G z\mathcal{R}$, the action is expressed with the conformal time η as

$$S_2 = \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x \left[v'^2 - c_s^2(\partial v)^2 + \frac{z''}{z}v^2\right]$$

which is equivalent to an action of a free scalar field with a time-dependent mass squared $\frac{z''}{z} = (aH)^2\left[(2 - \varepsilon_H - s + \frac{\eta_H}{2})(1 - s + \frac{\eta_H}{2}) - \frac{\dot{s}}{H} + \frac{2\dot{\eta}_H}{2H}\right]$

$$s \equiv \frac{\dot{c}_s}{Hc_s}, \quad \eta_H \equiv \frac{\dot{\varepsilon}_H}{H\varepsilon_H}$$

$$S_2 = \frac{1}{2}\int d\eta d^3x \left[v'^2 - c_s^2(\partial v)^2 + \frac{z''}{z}v^2\right] \qquad s \equiv \frac{\dot{c}_s}{Hc_s}, \quad \eta_H \equiv \frac{\dot{\varepsilon}_H}{H\varepsilon_H}$$

$$\frac{z''}{z} = (aH)^2 \Big[\underbrace{(2-\varepsilon_H - s + \frac{\eta_H}{2})(1 - s + \frac{\eta_H}{2}) - \frac{\dot{s}}{H} + \frac{\dot{\eta}_H}{2H}}_{\text{small slow-variation parameters}} \Big] \equiv (aH)^2(2+q)$$

☆ Using the de Sitter scale factor $a = -\frac{1}{H\eta}$, the normalized mode function reads

$$v_k = \left(-\frac{\pi\eta}{4}\right)^{1/2} H_\nu^{(1)}(-kc_s\eta) \cong \frac{1}{\sqrt{2kc_s}}\left(1 - \frac{i}{kc_s\eta}\right)e^{-ikc_s\eta} \quad \nu = \frac{3}{2}\left(1+\frac{4}{9}q\right)^{\frac{1}{2}} \cong \frac{3}{2}$$

☆ It behaves similarly to a massless scalar field in de Sitter background, so that long-wave nearly scale-invariant fluctuations will be generated.

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{4\pi k^3}{(2\pi)^3}|\mathcal{R}_k|^2 = \frac{4\pi k^3}{(2\pi)^3}\left|\frac{v_k}{z}\right|^2 = \frac{H^2}{8\pi^2 M_G^2 c_s \varepsilon_H}$$

evaluated at the sound horizon crossing $-kc_s\eta = 1$

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{4\pi k^3}{(2\pi)^3}|\mathcal{R}_k|^2 = \frac{4\pi k^3}{(2\pi)^3}\left|\frac{v_k}{z}\right|^2 = \frac{H^2}{8\pi^2 M_G^2 c_s \varepsilon_H}$$

★ The spectral index of the curvature perturbation is given by

$$\boxed{n_s - 1 \equiv \frac{d\ln \mathcal{P}_{\mathcal{R}}(k)}{d\ln k} = -2\varepsilon_H - \eta_H - s} \qquad \varepsilon_H \equiv -\frac{\dot{H}}{H^2} \qquad s \equiv \frac{\dot{c}_s}{Hc_s}, \quad \eta_H \equiv \frac{\dot{\varepsilon}_H}{H\varepsilon_H}$$

★ In the canonical slow-roll inflation, using the slow-roll equations we find

$$\varepsilon_H = -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{M_G^2 H^2} = \frac{3\dot{\phi}^2}{2V} \underset{3H\dot{\phi} = -V'}{=} \frac{M_G^2}{2}\left(\frac{V'}{V}\right)^2 \equiv \varepsilon_V \qquad \eta_V \equiv M_G^2 \frac{V''}{V} \qquad \eta_H = -2\eta_V + 4\varepsilon_V$$

so

$$\boxed{n_s - 1 = -6\varepsilon_V + 2\eta_V}$$

★ The scale dependence of the spectral index, "Running"

$$\boxed{\frac{dn_s}{d\ln k} = 16\varepsilon_V\eta_V - 24\varepsilon_V^2 - 2\xi_V} \qquad \xi_V \equiv M_G^4 \frac{V'V'''}{V^2}$$

These are important observable quantities!

Tensor perturbation from inflation

- ⋆ We derive a second-order action for the tensor perturbation $h_{\mu\nu}$.
  It is wise to make use of the known results on GW in the Minkowski space.
  So we first study perturbation around $\eta_{\mu\nu}$ taking metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

  Taking the TT gauge $h_{00} = h_{0i} = 0,\ h^{\alpha}_{\alpha} = h^{i}_{i} = 0,\ h^{;j}_{ij} = 0$,
  the Ricci scalar reads up to the second order

$$R = h^{ij} h^{;\mu}_{ij,\mu} + \frac{3}{4} h^{ij,\mu} h_{ij,\mu} - \frac{1}{2} h^{ij,l} h_{jl,i}$$

- ⋆ The transformation from $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
  to $ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j\right] \equiv \tilde{g}_{\mu\nu} dx^\mu dx^\nu$ can be done
  by the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

- ⋆ The Ricci tensors in two conformal metrices are related as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \ln \Omega - g_{\mu\nu} g^{\sigma\tau} \nabla_\sigma \nabla_\tau \ln \Omega$$
$$+ 2\nabla_\mu \ln \Omega \nabla_\nu \ln \Omega - g_{\mu\nu} g^{\sigma\tau} \nabla_\sigma \ln \Omega \nabla_\tau \ln \Omega$$

★ Putting $\Omega = a(\eta)$, the Ricci scalar reads up to the second order

$$\tilde{R} = a^{-2}\left(R + 6\frac{a''}{a} - 3\frac{a'}{a}h^{ij}h'_{ij}\right)$$

★ Since we are interested in tensor perturbations in the inflaitonary Universe let us introduce a cosmological constant to drive inflation, to consider

$$\begin{aligned} S_2 &= \frac{M_G^2}{2}\int (\tilde{R} - 2\Lambda)\sqrt{-\tilde{g}}d^4x\Big|_{\text{2nd order}} \\ &= \frac{M_G^2}{8}\int d\eta d^3x a^2 (h^{i\prime}_j h^{j\prime}_i - h^i_{j,l}h^{j,l}_i) \end{aligned}$$

using the background eq. $\Lambda a^4 = 2aa'' - a'^2$

★ Introducing new variables $z_T \equiv a/2, u_{ij} \equiv M_G z_T h_{ij}$ the action reads

$$S_T = \frac{1}{2}\int d\eta d^3x \left[u'^2_{ij} - (\nabla u_{ij})^2 + \frac{a''}{a}u^2_{ij}\right]$$

which is equivalent to the action of a massless scalar field.

In the de Sitter background $a = -1/(H\eta)$, we find

$$u^A_{ij} = \left(-\frac{\pi\eta}{4}\right)^{1/2} H^{(1)}_{3/2}(-k\eta)e^A_{ij}(\boldsymbol{k}), \quad A = +, \times$$

as before.

★ Introducing new variables $z_T \equiv a/2, u_{ij} \equiv M_G z_T h_{ij}$
the action reads

$$S_T = \frac{1}{2}\int d\eta d^3x \left[u_{ij}'^2 - (\nabla u_{ij})^2 + \frac{a''}{a}u_{ij}^2\right]$$

which is equivalent to the action of a massless scalar field.

In the de Sitter background $a = -1/(H\eta)$, we find

$$u_{ij}^A = \left(-\frac{\pi\eta}{4}\right)^{1/2} H_{3/2}^{(1)}(-k\eta)e_{ij}^A(\boldsymbol{k}), \quad A = +, \times$$

as before.

★ In case $\varepsilon_H \neq 0$, we can express $a = -\frac{1}{H_*\eta_*}\frac{1}{1-\varepsilon}\left(\frac{-\eta}{-\eta_*}\right)^{\frac{1}{2}-\nu_T}, \quad \nu_T = \frac{3}{2}\frac{1-\varepsilon/3}{1-\varepsilon}$

$$u_{ij}^A = \left(-\frac{\pi\eta}{4}\right)^{1/2} H_{\nu_T}^{(1)}(-k\eta)e_{ij}^A(\boldsymbol{k}), \quad A = +, \times$$

$e_{ij}^A(\boldsymbol{k})$ is the polarization tensor with $\langle e_{ij}^A(\boldsymbol{k})e^{*ijB}(\boldsymbol{k})\rangle = \delta^{AB}$

★ The power spectrum reads

$$\mathcal{P}_T(k) \equiv \frac{4\pi k^3}{(2\pi)^3}h_{ij}h^{*ij} = \frac{4\pi k^3}{(2\pi)^3}\frac{u_{ij}^A u^{*ijA}}{M_G^2 z_T^2} = \frac{2H^2}{\pi^2 M_G^2}$$

★ Tensor spectral index

$$\boxed{n_t \equiv \frac{d\ln \mathcal{P}_T(k)}{d\ln k} = -2\varepsilon_H}$$

★ The tensor-to-scalar ratio

$$\boxed{r \equiv \frac{\mathcal{P}_T(k)}{\mathcal{P}_\mathcal{R}(k)}} \qquad \frac{\mathcal{P}_T(k) = \dfrac{2H^2}{\pi^2 M_G^2}}{\mathcal{P}_\mathcal{R}(k) = \dfrac{H^2}{8\pi^2 M_G^2 c_s \varepsilon_H}}$$

$$\Downarrow$$

$$\boxed{r = 16 c_s \varepsilon_H = -8 c_s n_t}$$ consistency relation

Some of the recent particle physics models of inflation

stringy

phenomenological

★ KK … 3)

★ DE … l extra dimension

(Silverstein & Tong 04)

V … possible.

… ohyama & Kinoshita 07)

★ Ra … , Kallosh, Linde &Quevedo 04)

… uning of parameters

★ Ax … vity

Newer models of "stringy" inflation

★ Monodoromy model

$r = 0.03$ (Silverstein & Westphal 08)

★ Fibre inflation

$r = 0.005$ (Cicoli, Burgers, & Quevedo 08)

★ Warped Wilsonline DBI inflation

$r \sim 0.1$ (Avgoustidis & Zavala 08)

etc

★ Higgsflation (Kaloper, Sorbo & JY 08)

$r = 0.003$ detectable tensor perturbation even if variation in ϕ is small

★ Chaotic inflation in supergravity with shift symmetry

(Kawasaki, Yamaguchi & Yanagida 00)

$r = 0.15$ allows large variation of ϕ thanks to the shift symmetry

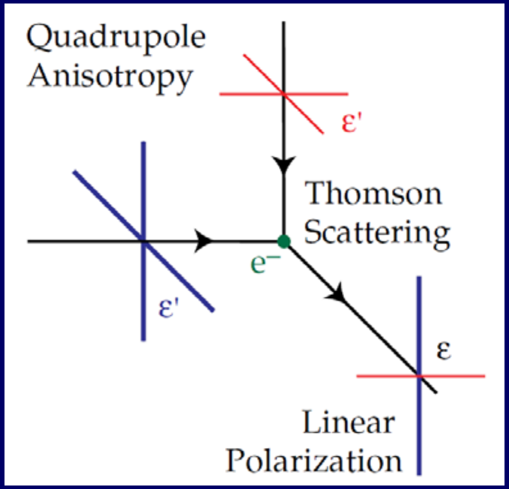
★ MSSM inflation (Allahverdi, Enqvist, Garcia-Bellido & Mazumdar 06)

$r \sim 10^{-25}$

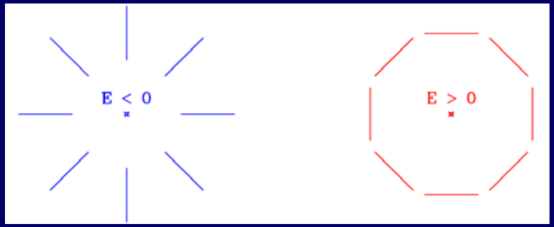
The inflaton can be identified in MSSM w/ fine-tuned parameters.

It can occur only if previous inflation with a slightly higher energy scale is realized with a sufficiently low reheat temperature, very contrived. (Kamada & JY 09)

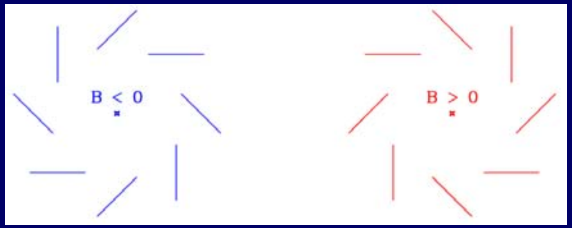
Low frequency components may be observed by B-mode polarization of CMB anisotropy



E mode



B mode



- Polarization is generated by quadrupole temperature anisotropy.
- E-mode from both scalar (density) and tensor perturbations.
- B-mode only from tensor perturbations.

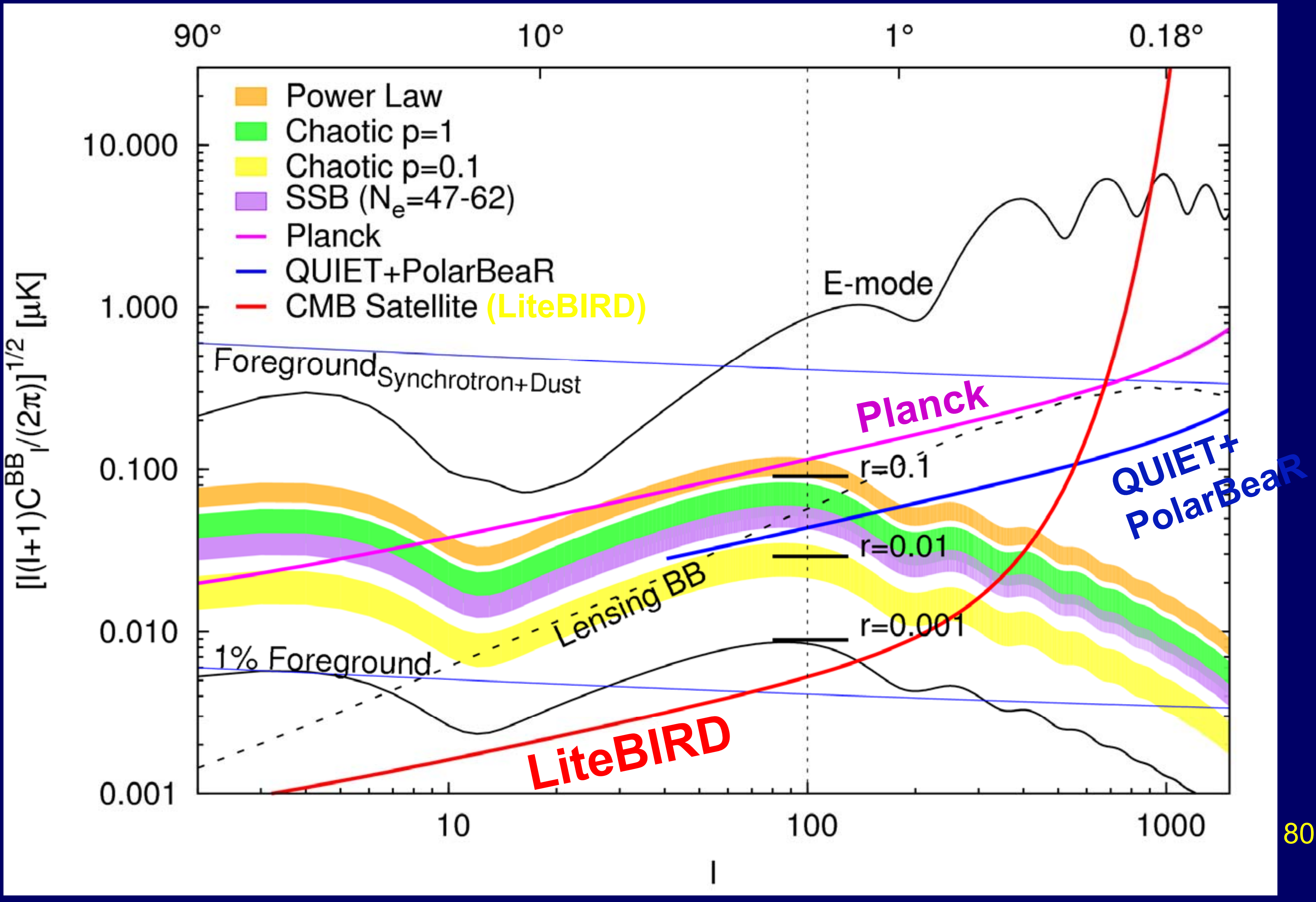
Planned or ongoing experiments and their expected sensitivity

PLANCK	$r \sim 0.1$	QUIET+PolarBear	$r \sim 0.01$
EPIC	$r \sim 0.001$	LiteBIRD	$r \sim 0.001$
CMB-POL	$r \sim 0.001$		

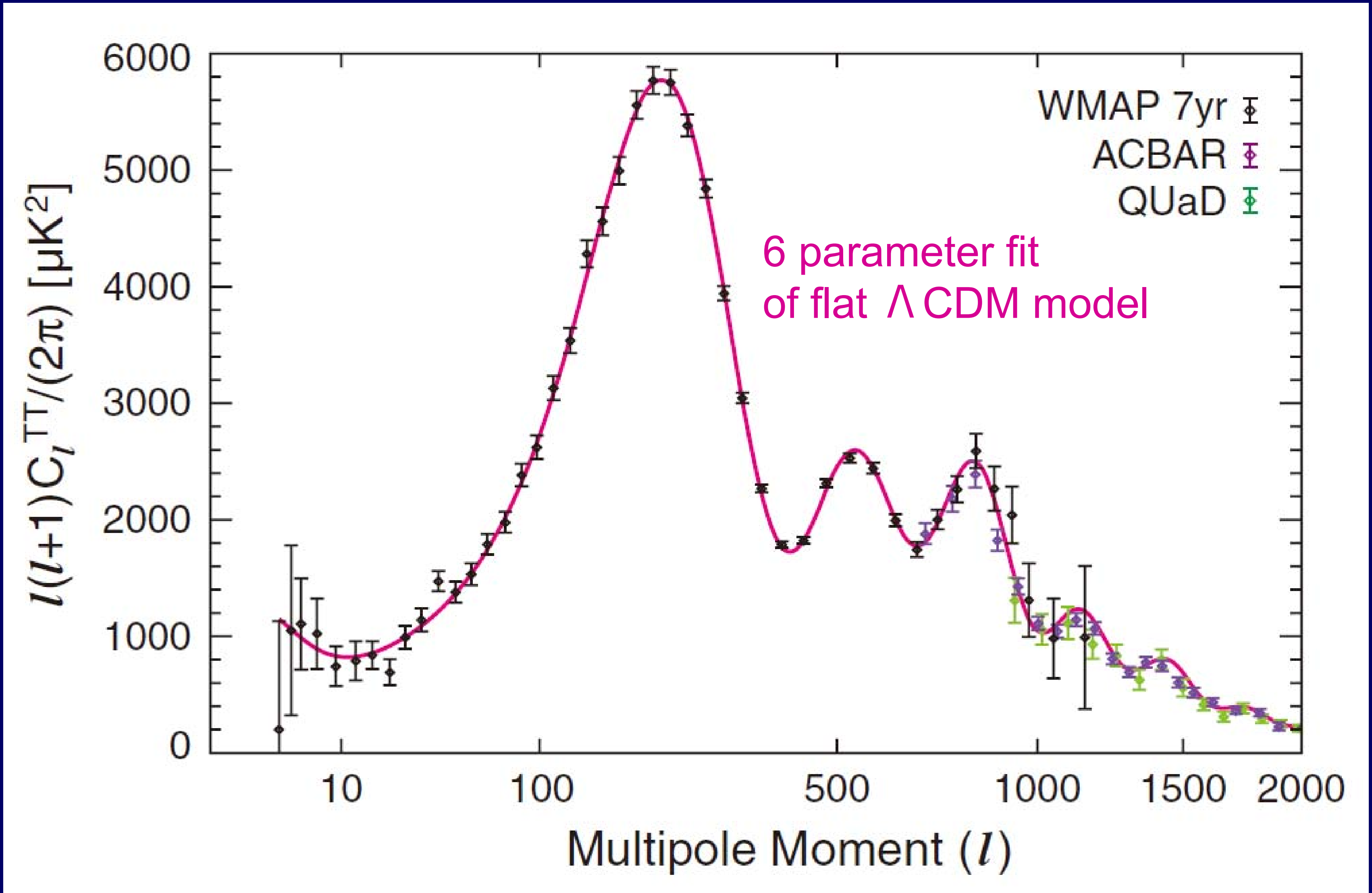
WMAP7 $r < 0.25$

Just one summary plot !

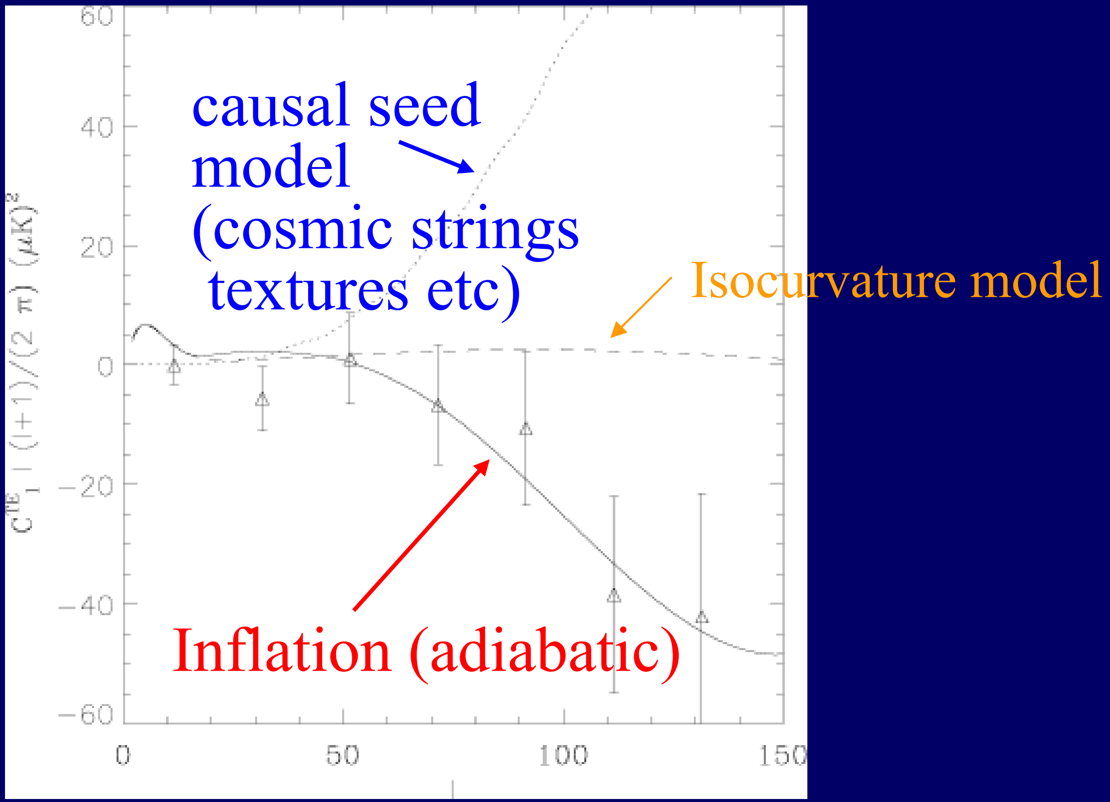
by M. Hazumi



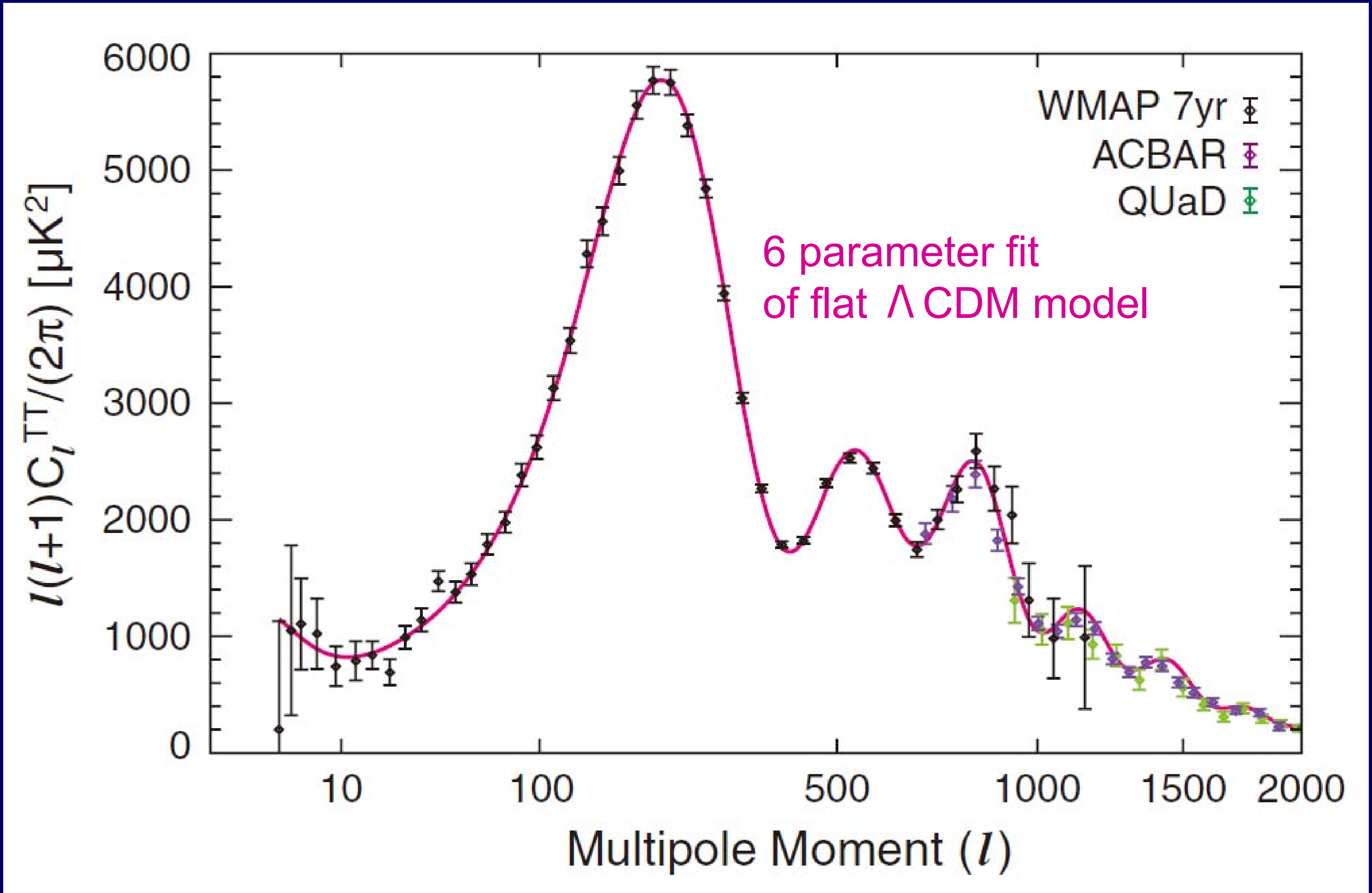
Theory and observations basically agree.



WMAP observed negative correlation between temperature anisotropy and E-mode polarization which is predicted by super-Hubble adiabatic fluctuations produced during inflation.



Theory and observations basically agree.



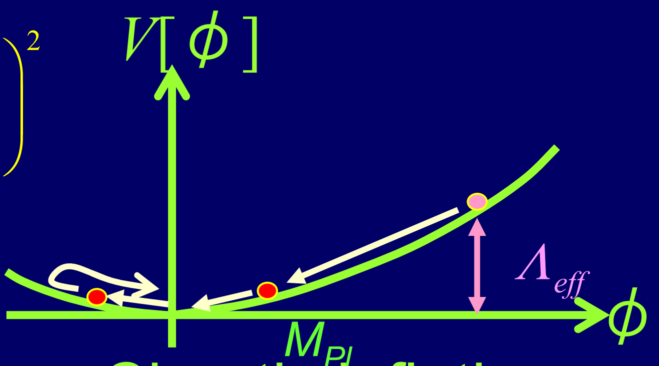
We wish to proceed model selection of inflation...

Observables: Large−field model

$$V[\phi] = \frac{1}{2} m^2 \phi^2$$

- Slow-roll parameters $\varepsilon_V = \eta_V = 2\left(\frac{M_G}{\phi}\right)^2 \quad \xi_V = 0$
- Number of e-folds from $\phi = \phi_N$ to the end of inflation

$$N = \int H dt = \int H \frac{d\phi}{\dot{\phi}} = \int \frac{3H^2}{V'[\phi]} d\phi = \int \frac{V[\phi] d\phi}{M_G^2 V'[\phi]} = \frac{1}{4}\left(\frac{\phi_N}{M_G}\right)^2$$



Chaotic inflation

- Amplitude of fluctuations

$$\mathcal{P}_{\mathcal{R}}(k_0) = \frac{H^2}{8\pi^2 M_G^2 \varepsilon_V} = \frac{1}{6\pi^2}\left(\frac{mN}{M_G}\right)^2 = 2.4 \times 10^{-9}$$

$$\Rightarrow m = 1.6 \times 10^{13}\,\mathrm{GeV}, \ \lambda < 8 \times 10^{-13} \ \text{for} \ \frac{\lambda}{4}\phi^4. \qquad @\, N \cong 55$$

- The coupling between the inflaton and other fields must be small.

  e.g. Yukawa coupling $h < 10^{-3}$, decay width $\Gamma_\phi = \frac{h^2}{8\pi} m < 6 \times 10^5\,\mathrm{GeV}$

$$T_R = 0.1 \left(\frac{200}{g_*}\right)^{1/4} \sqrt{M_{Pl}\Gamma_\phi} \cong 10^{11} \left(\frac{200}{g_*}\right)^{1/4} \left(\frac{\Gamma_\phi}{10^5\,\mathrm{GeV}}\right)^{1/2} \mathrm{GeV}$$

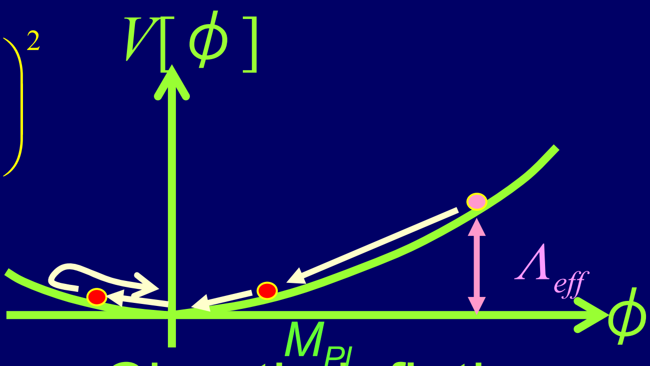
Observables: Large−field model

$$V[\phi] = \frac{1}{2} m^2 \phi^2$$

★ Slow-roll parameters $\varepsilon_V = \eta_V = 2\left(\frac{M_G}{\phi}\right)^2 \quad \xi_V = 0$

★ Number of e-folds from $\phi = \phi_N$ to the end of inflation

$$N = \int H dt = \int H \frac{d\phi}{\dot{\phi}} = \int \frac{3H^2}{V'[\phi]} d\phi = \int \frac{V[\phi] d\phi}{M_G^2 V'[\phi]} = \frac{1}{4}\left(\frac{\phi_N}{M_G}\right)^2$$



Chaotic inflation

★ Spectral index and its scale dependence

$$n_s = 1 - \frac{2}{N} = 0.964, \qquad \frac{dn_s}{d\ln k} = -6.6 \times 10^{-4}.$$

$$@\, N \cong 55$$

★ Tensor-to-scalar ratio

$$r = 16\varepsilon_V = 0.15.$$

Observable by Planck!

Observables: Small–field model

$$V[\phi] = \frac{\lambda}{4}(\phi^2 - v^2)^2 \qquad v \equiv \beta M_G$$

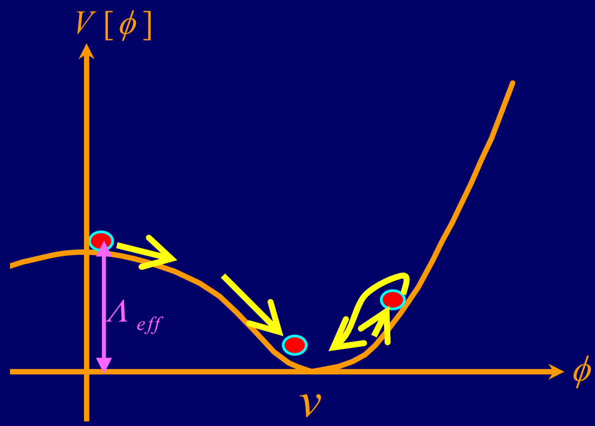
- Slow-roll parameters

$$\varepsilon_V = \frac{8M_G^2\phi^2}{(\phi^2 - v^2)^2}, \quad \eta_V = \frac{4M_G^2(3\phi^2 - v^2)}{(\phi^2 - v^2)^2}, \quad \xi_V = \frac{96M_G^4\phi^2}{(\phi^2 - v^2)^3}$$

- Inflation ends when $\varepsilon_H = \varepsilon_V = 1$ at the field value

n
f
l

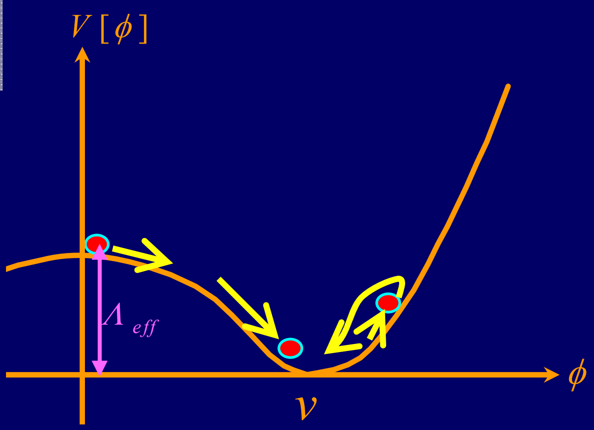
$$\phi_f^2 = v^2 + 4M_G^2 - \sqrt{16M_G^2 + 8M_G^2 v^2}$$
$$\cong (\beta^2 - 2\sqrt{2}\beta)M_G^2$$



- Number of e-folds from $\phi = \phi_N$ to the end of inflation

$$N = \int_{\phi_N}^{\phi_f} H\frac{\mathrm{d}\phi}{\dot{\phi}} = \frac{\beta^2}{4}\ln\frac{\phi_f}{\phi_N} - \frac{1}{8M_G^2}(\phi_f^2 - \phi_N^2) \simeq \frac{\beta^2}{4}\left(\ln\frac{\phi_f}{\phi_N} - \frac{1}{2}\right)$$

Observables: Small–field model



$$V[\phi] = \frac{\lambda}{4}(\phi^2 - v^2)^2 \qquad v \equiv \beta M_G$$

- ★ Curvature perturbation

$$\mathcal{P}_{\mathcal{R}}(k_0) = \frac{\lambda(\phi_N^2 - v^2)^4}{768\pi^2 M_G^6 \phi_N^2} \simeq \frac{\lambda\beta^8}{768\pi^2}\left(\frac{M_G}{\phi_N}\right)^2 \simeq \frac{\lambda\beta^8}{768\pi^2(\beta-\sqrt{2})^2}\exp\left(\frac{8N}{\beta^2}+1\right)$$

Taking $N = 55, \;\; \beta = 15$, the normalization gives $\lambda = 7\times 10^{-14}$.

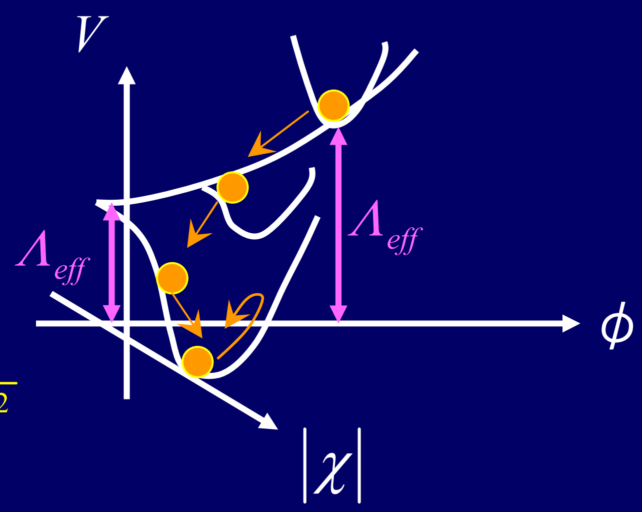
- ★ Spectral index and its scale dependence

$$n_s - 1 = -\frac{8(3\phi_N^2 + v^2)M_G^2}{(\phi_N^2 - v^2)^2} \simeq -\frac{8}{\beta^2}, \qquad 0.964$$

$$\frac{\mathrm{d}n_s}{\mathrm{d}\ln k} = -\frac{(320v^2\phi_N^2 + 192\phi_N^4)M_G^4}{(\phi_N^2 - v^2)^4} \simeq -\frac{320}{\beta^6}\left(\frac{\phi_N}{M_G}\right)^2, \qquad \text{tiny}$$

Observables: Hybrid inflation model

$$V[\phi] = V_0 + \frac{m^2}{2}\phi^2$$ near the origin



- Consider false-vacuum dominated case

$$\varepsilon_V = \frac{M_G{}^2}{2}\left(\frac{m^2\phi}{V_0}\right)^2 = \frac{1}{18}\left(\frac{m}{H}\right)^4\left(\frac{\phi}{M_G}\right)^2 \qquad \eta_V = \frac{M_G^2 m^2}{V_0} = \frac{m^2}{3H^2}$$

- Spectral index and its scale dependence

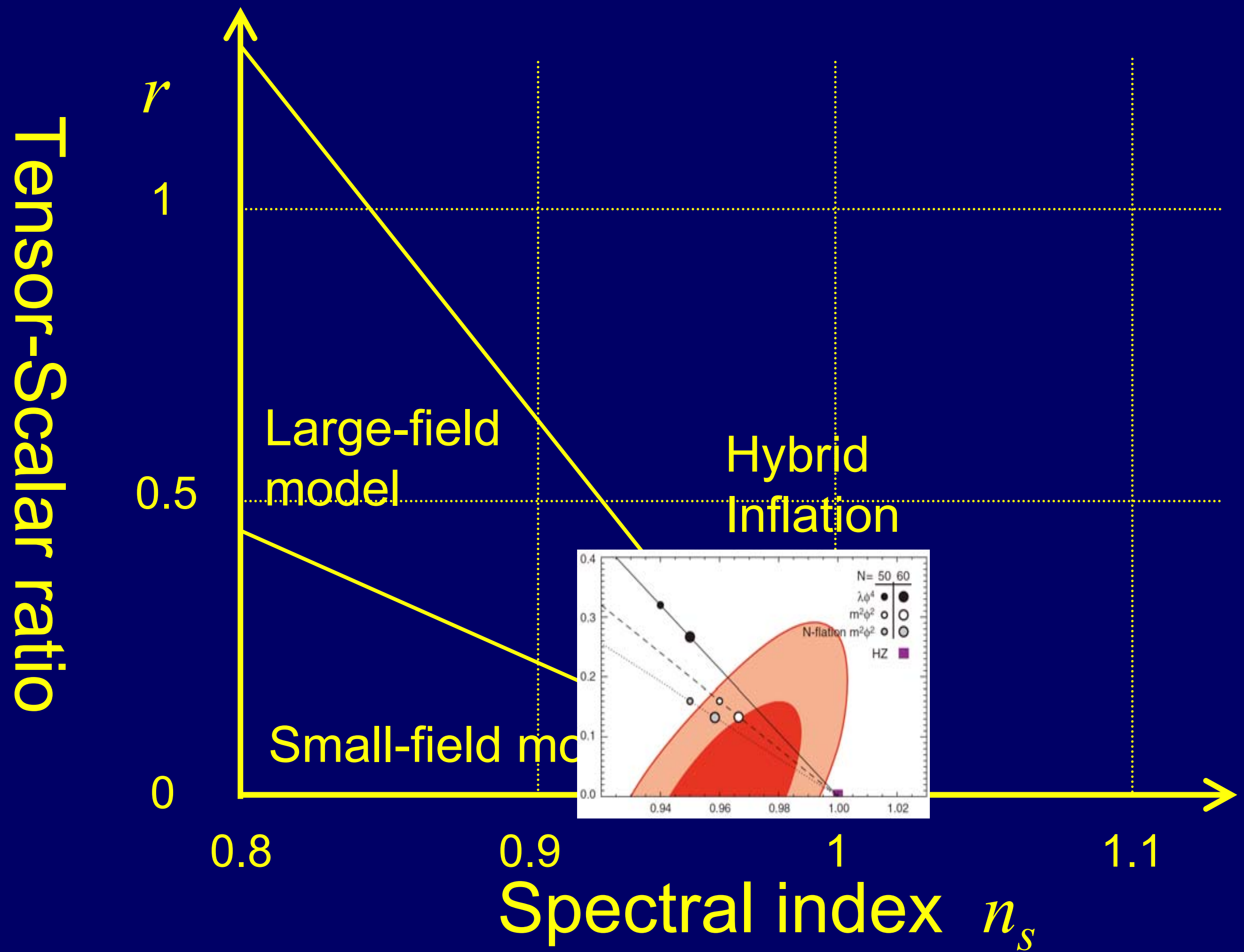
$$n_s - 1 \cong 2\eta_V = \frac{2m^2}{3H^2}$$

$$\frac{dn_s}{d\ln k} \cong 16\varepsilon\eta = \left(\frac{2m^2}{3H^2}\right)^3\left(\frac{\phi}{M_G}\right)^2 = (n-1)^3\left(\frac{\phi}{M_G}\right)^2 \sim 10^{-4}\left(\frac{\phi}{M_G}\right)^2$$

- Tensor-to-scalar ratio

$$r = 2\left(\frac{2m^2}{3H^2}\right)^2\left(\frac{\phi}{M_G}\right)^2 = (n-1)^2\left(\frac{\phi}{M_G}\right)^2 \sim 0.005\left(\frac{\phi}{M_G}\right)^2$$

Inflation models may be distinguished by observations.



$WMAP+\mathrm{BAO}+H_0$

(Dodelson, Kinney, Kolb 97)

Deviation from Gaussian: NonGaussianity of fluctuations

may also help distinguish models.

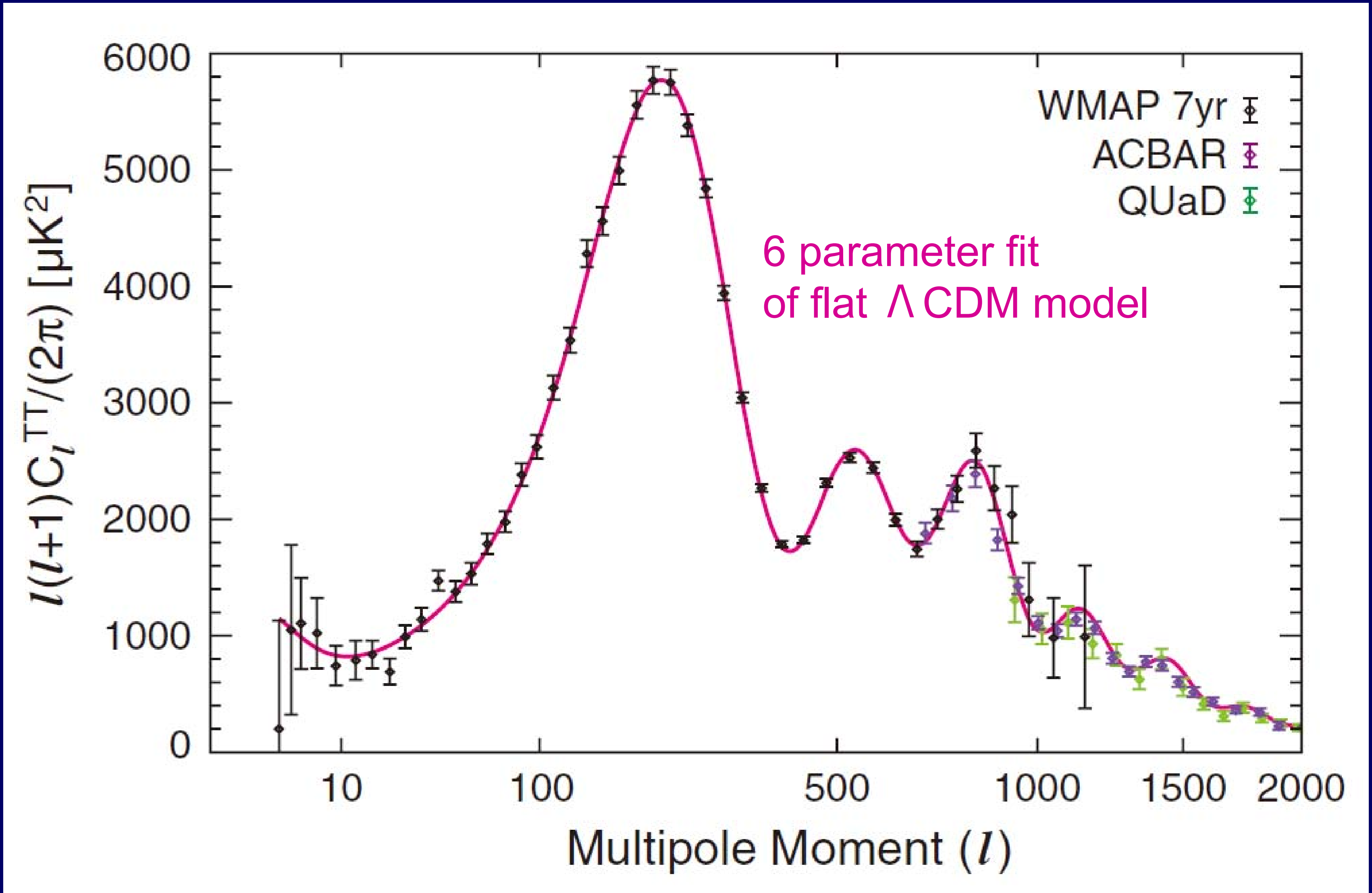
- ⋆ Potential-driven slow-roll models
  NonGaussianity is small, because the inflaton is very weakly coupled with other fields as we have seen.

- ⋆ k-inflation, G-inflation,…
  NonGaussianity can be large.

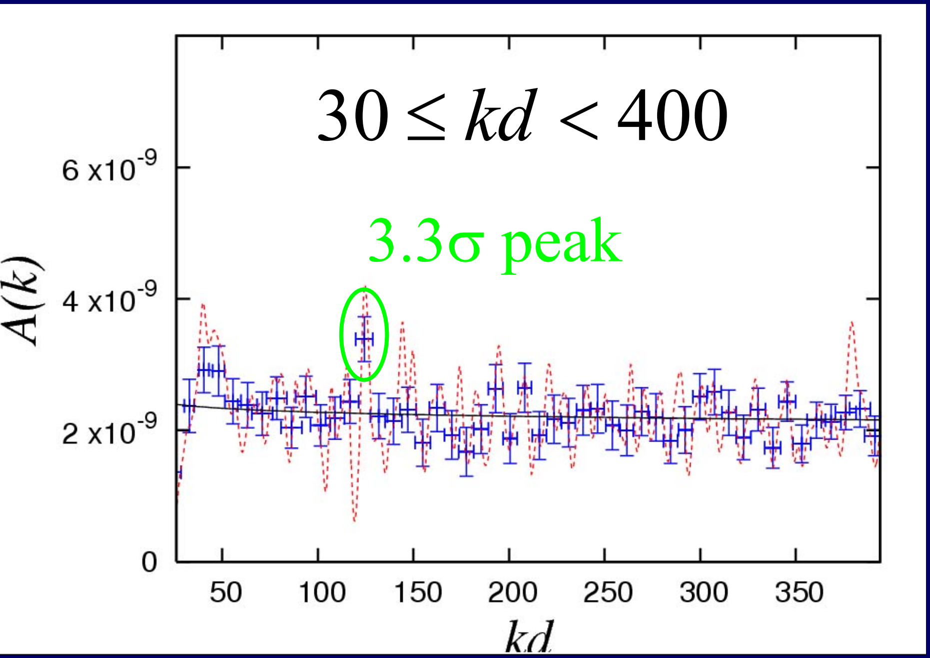
- ⋆ Beyond the single-field inflation

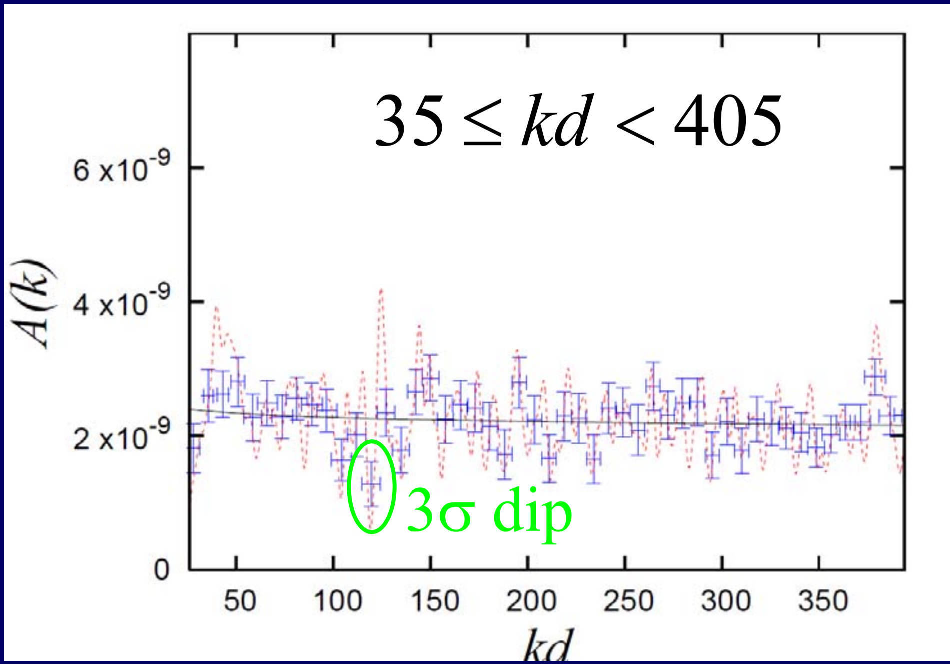
NO detection yet

Theory and observations basically agree.

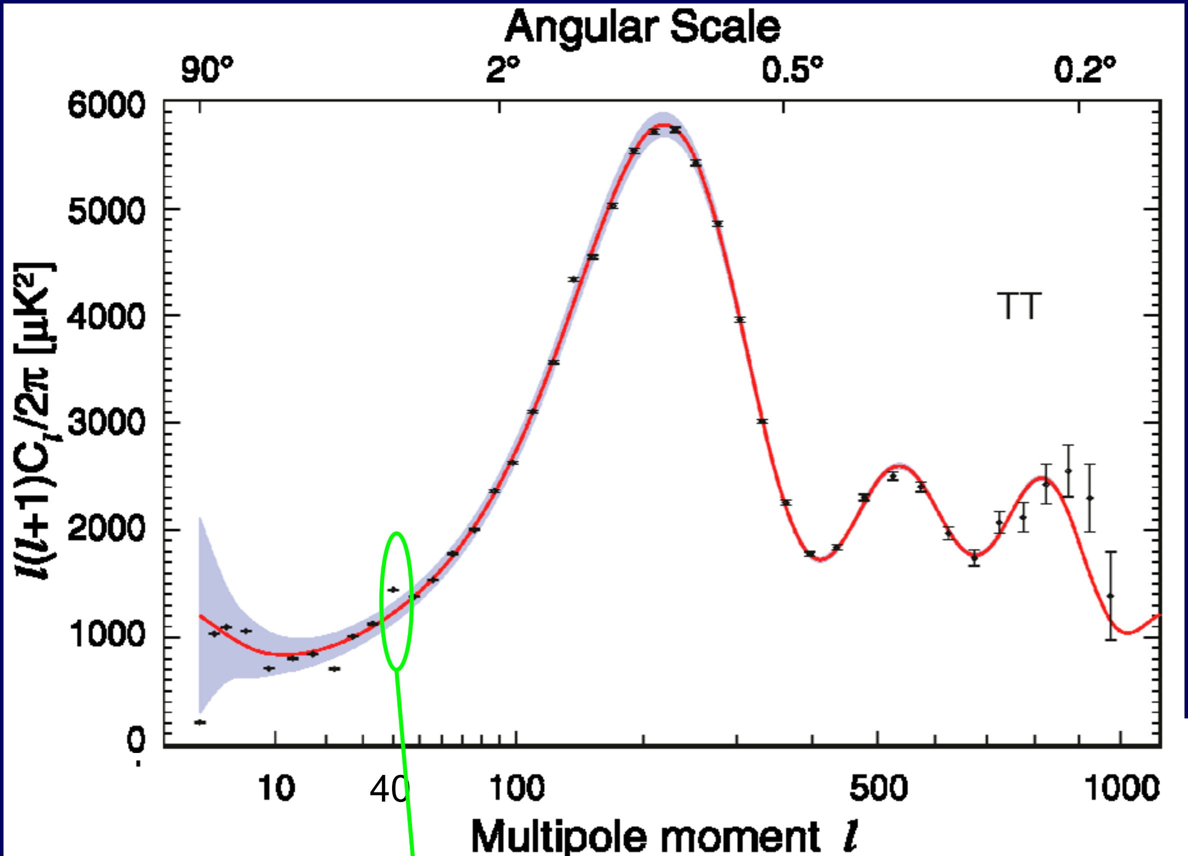


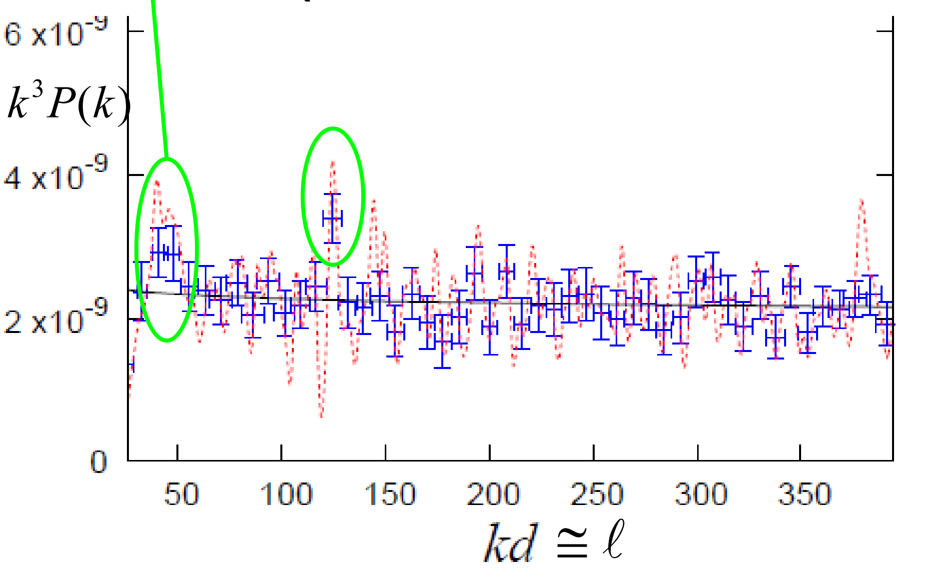
If you look at it closer in detail...





In fact, if we change the wavenumber domain of decomposition slightly, we obtain a dip rather than an excess even for the band power analysis.





Deviation around $kd \approx \ell \approx 40$ can be seen even in the binned C_ℓ but those at 125 can not be seen there.

(Nagata & JY 08)

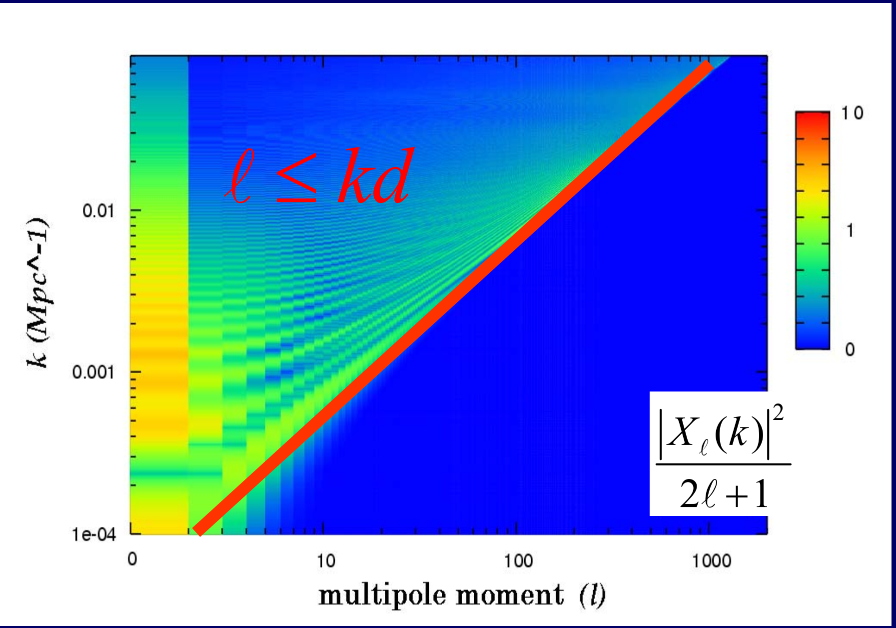
Forward Analysis

- Assume various shapes of modified power spectrum $P(k)$ with three additional parameters in addition to the standard power-law.

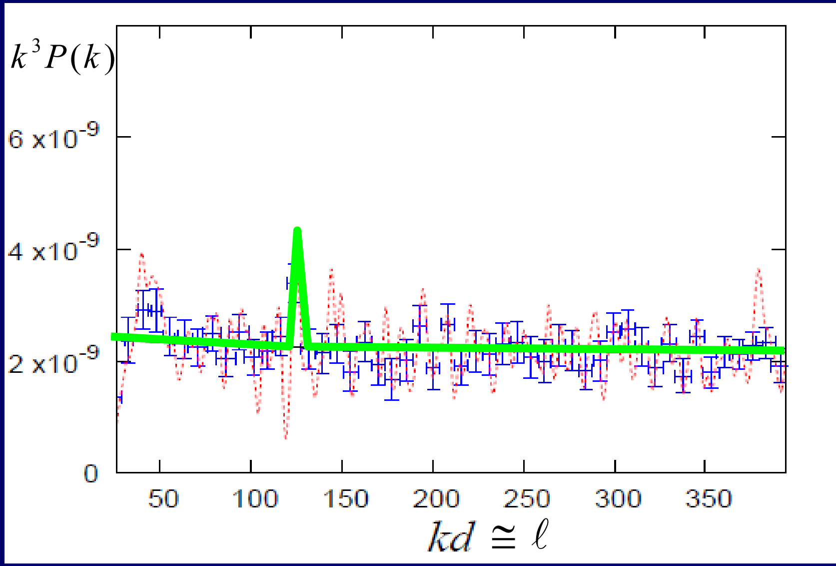
- Perform Markov-Chain Monte Carlo analysis with CosmoMC with these three additional parameters in addition to the standard 6 parameter ΛCDM model.

Transfer function shows that C_ℓ depends on $P(k)$ with $kd \geq \ell$.

$$C_\ell = \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{(2\ell+1)^2} |X_\ell(k)|^2 P(k)$$



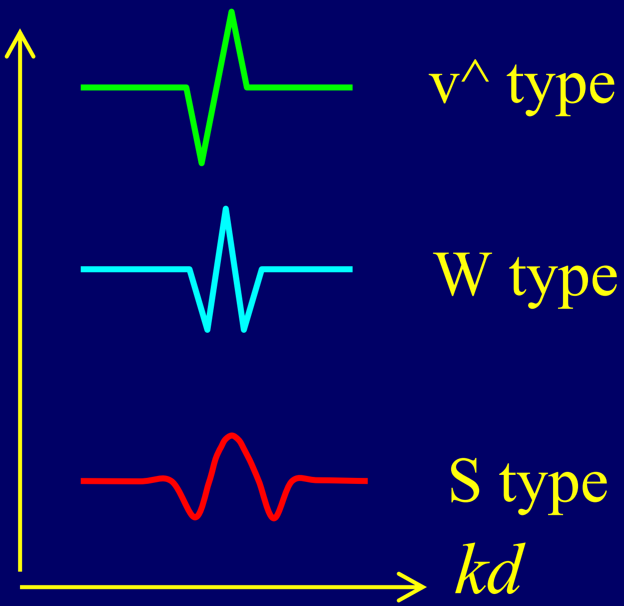
If we add some extra power on $P(k)$ at $kd \approx 125$, it would modify all C_ℓ's with $\ell \leq kd \approx 125$.

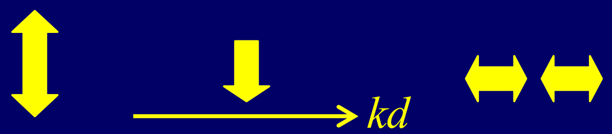


Simply adding an extra power around $kd \approx 125$ does not much improve the likelihood, because it modifies the successful fit of power-law model at smaller ℓ's.

Consider power spectra which change C_ℓ's only locally.

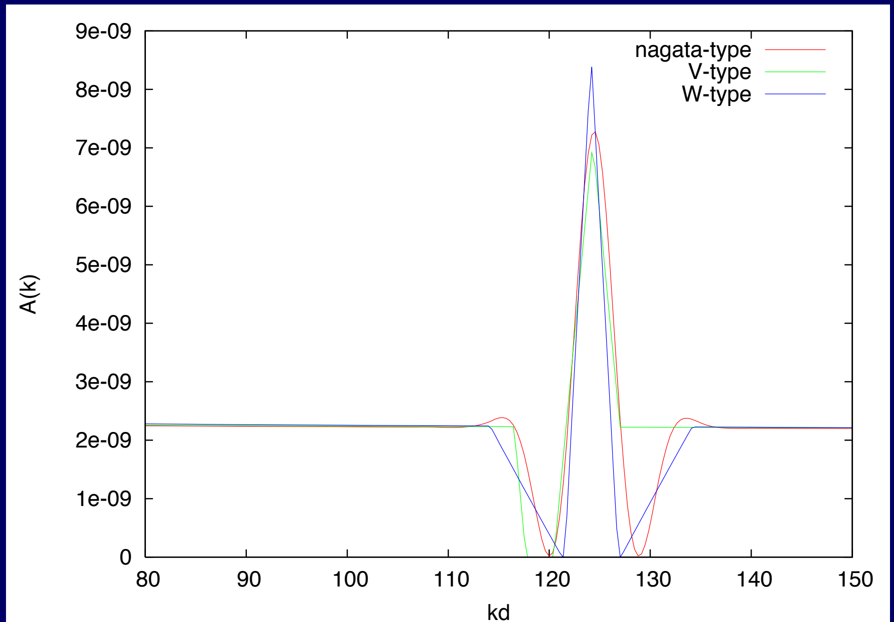
$A(k) \equiv k^3 P(k)$





Height, location, & width of the peak are 3 additional parameters.

$$k^3 P(k) = A\left(\frac{k}{k_0}\right)^{n-1} + B\left(\frac{k}{k_0}\right)^{n-1} \exp\left(-\frac{(k-k_*)^2}{\kappa^2}\right)\cos\left(\pi\frac{k-k_*}{\kappa}\right)$$



χ^2_{eff} improves as much as 22 by introducing 3 additional parameters.

	Power law	Λ-type	V^{Λ}-type	S-type	W-type
Ω_b	0.0438	0.0441	0.0443	0.0441	0.0444
Ω_m	0.256	0.256	0.260	0.257	0.262
Ω_Λ	0.744	0.744	0.740	0.743	0.738
H_0	72.1	72.1	71.7	72.0	71.6
$10^{10}A$	23.88	23.24	23.51	23.34	23.90
n_s	0.964	0.975	0.969	0.970	0.964
τ	0.0864	0.0879	0.0846	0.0835	0.0845
$\Delta\chi^2_{\rm eff}$	0	-6.5	-19	-22	-16
$k_* d$		124.5	124.4	124.5	$\cdots$
$10^{10}B$		23.80	47.26	55.66	37.95
$-2\ln\mathcal{L}$	2658.1	2651.6	2639.1	2636.2	2641.8
D.o.F.		3	3	3	3

(Ichiki, Nagata, JY, 08)

If χ^2 improves by 2 or more, it is worth introducing a new parameter, according to Akaike's information criteria (AIC).

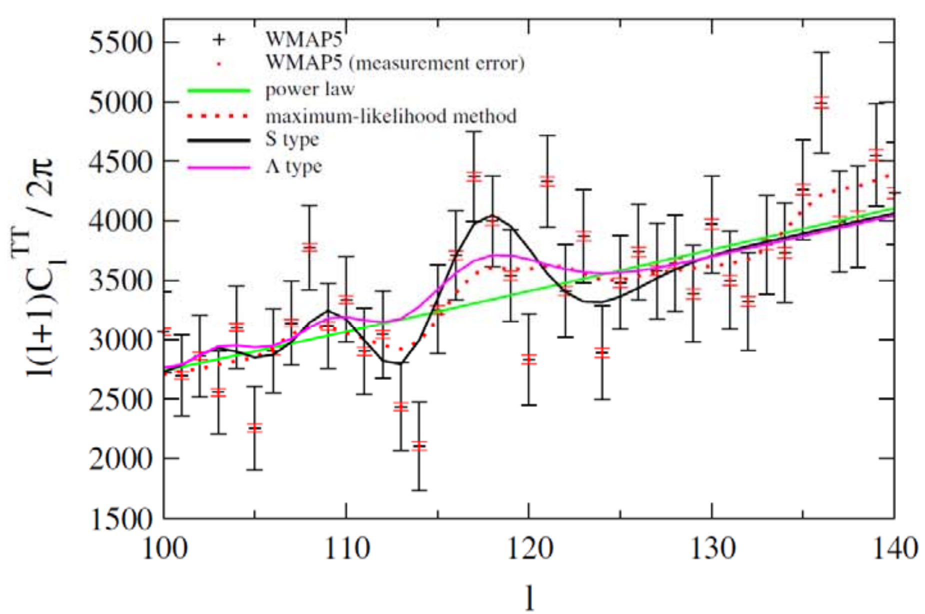
Unlike our reconstruction methods, MCMC calculations use not only TT data but also TE data.

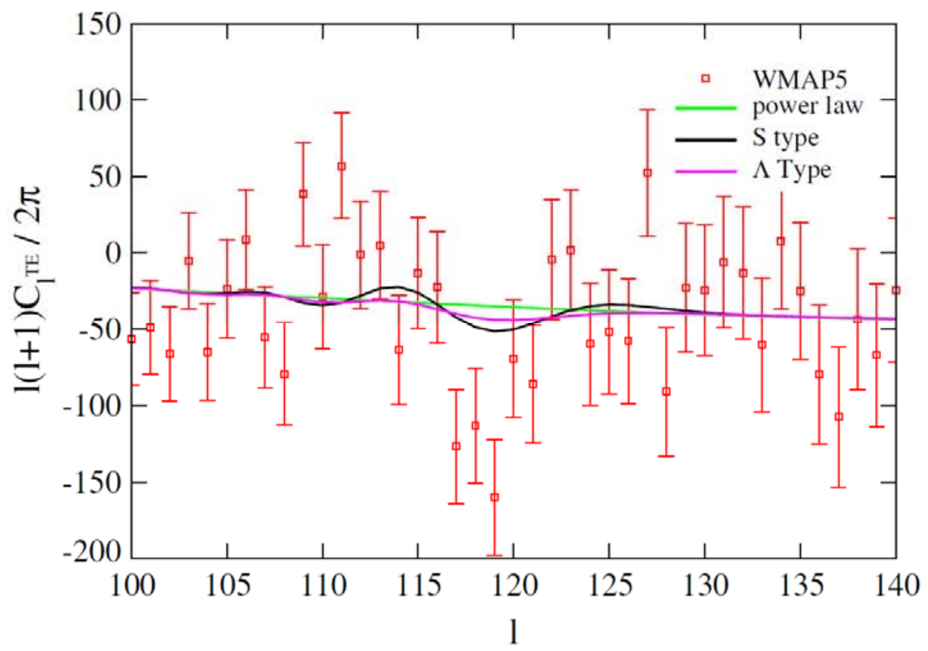
$$\Delta\chi^2_{eff} \text{ due to improvement of TT fit} = -12.5$$
$$\Delta\chi^2_{eff} \text{ due to improvement of TE fit} = -8.5$$

It is intriguing that our modified spectra improve TE fit significantly even if we only used TT data in the beginning.

TT(temp-temp) data and model

TE(temp-Epol) data and model





- Posterior probability to find vanishingly small deviation from a power-law.

$$P(B < 1 \times 10^{-10}) = 4.8 \times 10^{-5}.$$

based on a local analysis in the range $\Delta kd = 20$.

- Posterior probability to find vanishingly small deviation from a power-law at any observed wavenumber domain.

$$P(B < 1 \times 10^{-10}) \sim 8 \times 10^{-4}.$$

based on a global analysis in the range $40 < kd < 380$.

- This may or may not be so by chance.
  In either case, however,…

The presence of such a fine structure changes the estimate of other cosmological parameters at an appreciable level by Planck.

Maximum of the difference from the power law

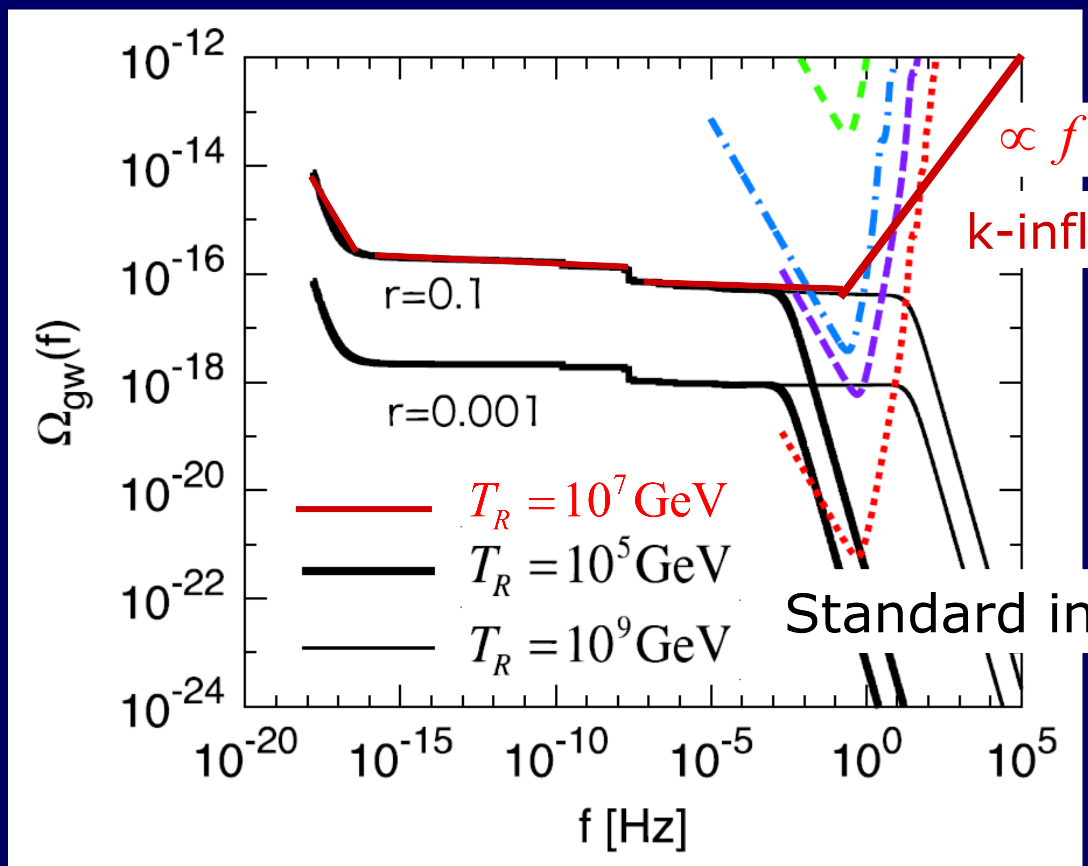
	Power law	Λ-type	V^{Λ}-type	S-type	W-type	$\Delta_{\max}$	WMAP5	Planck
Ω_b	0.0438	0.0441	0.0443	0.0441	0.0444	0.0006	0.0030	0.0003
Ω_m	0.256	0.256	0.260	0.257	0.262	0.006	0.027	
Ω_Λ	0.744	0.744	0.740	0.743	0.738	0.006	0.015	0.009
H_0	72.1	72.1	71.7	72.0	71.6	0.5	2.7	2.7
$10^{10}A$	23.88	23.24	23.51	23.34	23.90	0.54	1.12	
n_s	0.964	0.975	0.969	0.970	0.964	0.006	0.015	0.0045
τ	0.0864	0.0879	0.0846	0.0835	0.0845	0.0029	0.017	0.005
$\Delta\chi^2_{\rm eff}$	0	-6.5	-19	-22	-16			
$k_* d$		124.5	124.4	124.5	$\cdots$			
$10^{10}B$		23.80	47.26	55.66	37.95			
$-2\ln\mathcal{L}$	2658.1	2651.6	2639.1	2636.2	2641.8			
D.o.F.		3	3	3	3			

Expected Errors by PLANCK

Higher frequency tensor perturbation

Its spectrum can be used to probe post-inflationary thermal history of the early Universe.

sensitivity curves of DECIGO



$w = 1$ between inflation and reheating

$w = 0$ between inflation and reheating

Conclusion

The precision cosmology is entering a new era with even higher precision.

Hopefully we will be able to know which if any is the correct inflation model that occurred at the birth of our Universe.

上州無智亦無才
剛毅木訥易被欺
唯以正直接萬人
至誠依神期勝利

鑑三